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PRACTICAL
PLANE AND SOLID
GEOMETRY
FOR ELEMENTARY STUDENTS

BY

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PREFACE

MANY of our schools and colleges are now well equipped with laboratories in which experimental work involving quantitative measurements is regularly carried on as part of the ordinary school course, and it is coming to be recognised that elementary mathematics should be taught in relation to such work.

Thus in arithmetic the use of decimally subdivided scales greatly facilitates the acquiring of a knowledge of decimals. The rules of mensuration may be established along with actual measuring and weighing operations relating to lines, surfaces, and solids. Such measurements, together with measurements of angles—in a plane or in space—naturally involve elementary notions of Geometry, Algebra, and Trigonometry. The employment of graphs introduces, at an early period, the use of rectangular coordinates and equations to lines and curves. And experiments in elementary mechanics require some knowledge of vectors.

In its report on the teaching of mathematics, the Committee of the British Association, appointed in 1901, recommends that elementary mathematics should be taught in the manner above stated; and that geometry should take its proper place in the general scheme, the formal study of deductive geometry being preceded by an experimental course, based on careful drawing and accurate measurement, using straight-edge and squares, scales, and protractor. Teachers and examiners alike seem quite ready to accept this report, and are beginning to act upon it. The present volume is written in full sympathy with these views.

We have not assumed that the student has read Euclid. In a course of geometry based on measurement, mathematical instruments play an important part. The use of any mechanical device which may be an aid to draughtsmanship, such as that of tracing-paper, is to be encouraged. A good teacher will see to it that the appliances used in his classes are maintained in a state of efficiency, so that a student may learn to depend with confidence on the accuracy of his straight-edges, squares, and scales; just as in a good engineering workshop the workman must be able to rely on the plane-surfaces, machine tools, and gauges in the production of good work.

Again, geometry must not be isolated from other branches of mathematics, as if it were something quite apart and different in kind; each must be taught with due regard to the whole, giving to, and receiving from the others. It is one of the advantages of the proposed system that artificial barriers between the parts will to a large extent be removed.

Thus the solution of a problem may be effected partly by arithmetic and partly by drawing. Or results obtained graphically may be checked independently by arithmetical calculation. As soon as a youth is able to plot an angle and draw a right-angled triangle, and can measure lengths and work arithmetic, he may begin the compilation or verification of tables of sines, cosines, and tangents of angles, obtaining his first notions of trigonometry quite naturally in the drawing class. Subsequently, in problems solved by constructing and measuring right-angled triangles —and there are many such—the results can be verified by calculations, employing the trigonometrical tables. The student thus soon gets quite accustomed to the use of such tables, an acquirement of considerable value. Further, the experimental study of mechanics requires the aid of geometry; and, in return, geometry enters a region of the utmost importance, touching nature at many points, alive with interest, and potentially of very great educational value.

In the examples appended to the articles and chapters numerical answers are generally given, with the object of inducing students to acquire the habit of measuring results. A student is not expected to work all the examples, or even to make himself acquainted with every method of proceeding that may be explained in the text. Some good teachers will make little use of the examples provided, wisely preferring their own, or any which they may encourage their students to invent. The special examples relating to work in the laboratory are among the best, and these cannot be given in a general text-book.

In Solid Geometry the principal aim is that the reader shall become acquainted with the representation and measurement of lengths and angles in space. The subject may prove somewhat troublesome to a beginner. The method of treatment explained in the text is one which the author has found successful. The early problems are each worked in several fundamental ways which mutually illustrate one another, and the student makes models in further illustration. The introduction of the pictorial representation which the beginner himself draws to scale, has proved to be a distinct gain.

Models supplied to a student containing all the construction lines are of little value. But it is very desirable that two or three well-made models of lines and planes be provided in the laboratory or class-room, entirely devoid of such lines, and suitable for the measurement of lengths, angles, and dimensions in space, the measured data being afterwards transferred to the drawing-paper, and the original measurements checked and verified by methods of Descriptive Geometry.

The author has aimed at presenting and developing the subject-matter in such a way that the reader not only acquires a knowledge of Geometrical Principles, but is trained so as to be able to put his knowledge to practical use in the large and growing domain where graphical processes are proving so valuable. The student thus recognises the intrinsic importance of the subject, and his interest is stimulated and maintained.

Geometry is essentially a vector subject, and although it is well at the beginning to confine attention to the *magnitudes* of lines, angles, and figures, yet the introduction of the idea of a vector should not be too long delayed. The conception is natural and easy, and in its many applications geometry becomes a living subject. A notation for vectors has been adopted which brings into prominence the association of magnitude and direction, facilitates the setting of questions, renders it easy for a student to give complete answers, and is adapted for the alternative method of numerical computation, which may so often with advantage accompany graphical calculation.

Practical Geometry as taught in elementary schools at present suffers from being regarded too exclusively from an Art point of view. As instruction improves, and a better foundation is laid, there are future developments in store suspected by few

J. HARRISON.

ROYAL COLLEGE OF SCIENCE,
LONDON,
February 1903.

NOTE by PROFESSOR PERRY, M.E., D.Sc., F.R.S., Professor of Mechanics and Mathematics at the Royal College of Science, London.

IN my reply to the British Association discussion on the Teaching of Mathematics (p. 102) I say that a very great reform has been effected in Practical Geometry, "and that it is no longer a mere collection of rules; it is now really an educational subject allowing a student to let his mind and imagination develop in several of many directions, and bringing him into contact with many subjects through the common-sense application of a very few general principles." I can recommend this book of Mr. Harrison's to all who wish to assist in the reform.

JOHN PERRY.

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CHAPTER I

USE OF INSTRUMENTS

1. Mechanical appliances used in drawing.—In elementary geometry, the constructions are based almost entirely on the straight line and circle. In pure geometry a line, straight or curved, is conceived as having no breadth. In practical geometry, the lines actually drawn on paper must of necessity be possessed of a certain width, but we endeavour to draw them as fine, and also as truly straight, or circular, as is practicable.

In order to accomplish this, we require the aid of tools, and these tools should be accurately made and kept in proper working order. The following appliances are in common use :—

Drawing-paper and means of securing it.

A **drawing-board**, having a flat surface to support the paper. The edges of the board should be straight.

A **pricker** for locating points.

A **lead pencil** for drawing lines, with means of sharpening it.

Squares with their edges straight, to enable straight lines to be drawn, and to facilitate the drawing of systems of parallel and perpendicular lines.

Compasses for describing circles in pencil.

Dividers for stepping off distances.

Scales for measuring and setting off lengths.

A **protractor,** or a scale or **table of chords,** for measuring and plotting angles.

A little stout **tracing paper,** for the application of the method of superposition.

In order to be able to maintain a high standard of accuracy in his graphical work, a student should have access to the tools necessary to keep his instruments in a state of efficient repair, namely, a **joiner's bench** with **vice,** a **trying plane,** sand-paper, a fine oil-stone, and a thin machined **steel straight edge** about a yard long. Every school should possess these, and special instruction in their use may with advantage be given.

We will now describe in detail these various drawing instruments, and give illustrations of their use and tests of their accuracy. In this connection some valuable preliminary knowledge of geometry will be obtained.

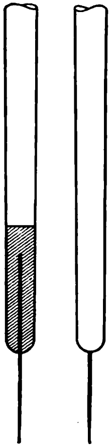
2. **Drawing-paper.**—Good hard cartridge-paper with a smooth white surface, answers every purpose. It is commonly fixed by drawing-pins, but these are apt to impede the movements of the squares, and as a substitute, fine tacks or sealing-wax may often be used with advantage.

3. **The drawing-board.**—A suitable size is "half imperial," 22″ × 15″. It is made of pine, a soft wood not very liable to warp, and "framed" at the ends, or with "battens" at the back. To true up its edges, should this be necessary, the board is secured in the vice and the trying plane is used. This requires some little skill in manipulation.

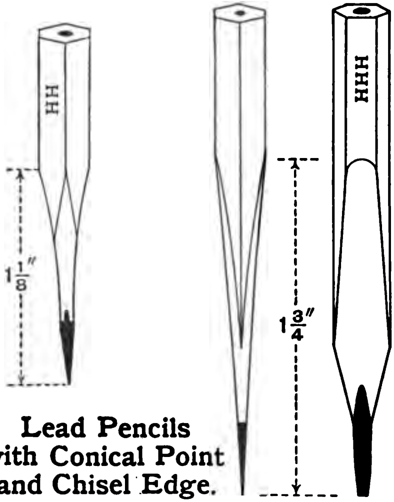
4. **The pricker and lead pencil.**—A needle mounted in a penholder makes a very good pricker. See the figure.

Note.—The hole in the penholder will require to be drilled. A needle ground to a triangular shaped section can be purchased and used as a drill.

The lead of the drawing pencil should be tolerably hard, say HH or HHH. A flat or **chisel edge** is stronger and more durable than a conical point. It is made as follows; see the figure:—Cut away the wood for a distance of about



Pricker.



Lead Pencils with Conical Point and Chisel Edge.

$1\frac{3}{4}''$ from the end, laying bare about $\frac{3}{8}''$ of lead. Then rub the lead over a piece of fine sand-paper, known as **glass paper**, so as to form the chisel edge as shown.

The student may also have an HH pencil with an ordinary **conical point**; this is useful in lettering diagrams, making calculations, etc. A point of this kind is also best finished by the use of the glass paper.

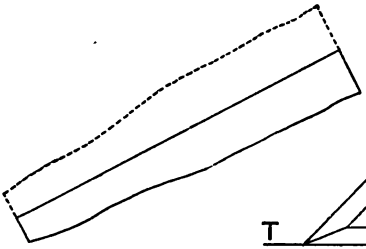
The student is **cautioned against** the use of **soft pencils.** An HB lead should never be used in graphical work. With an edge sufficiently fine it is too weak to bear the working pressure. Also the edge wears very rapidly, and the soft lead is apt to rub up and smear the drawing-paper in the vicinity of the lines.

5. The straight edge and the straight line.—To be able to draw a line which is sensibly straight is of fundamental importance in Practical Geometry. Without this, accuracy is impossible, and the work of little value; the student must give some attention to the matter. A fine thread stretched on the board would be straight. The beginner can readily make for himself a fairly good straight edge by merely folding carefully a piece of drawing-paper. In order to test its accuracy he might draw a line along its edge with a fine pencil; then turning the paper over on to its other face about the edge as a hinge, see whether a second line could now be drawn along the edge so as to coincide entirely with the first line. See the figure.

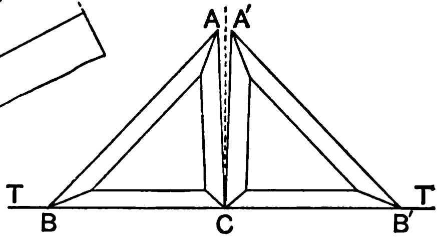
But in practical work the draughtsman relies on the edges of his tee and set-squares in drawing straight lines. These edges should be tested from time to time in the manner just described, or preferably, against a standard steel straight edge, if one is available. Any defects which may be discovered should be remedied by the use of a trying plane. The latter may be turned face upwards, and the edge of the square passed over the cutting blade, care being taken to hold the plane of the square perpendicular to the face of the plane, and to avoid bending the square by the exertion of undue pressure.

6. The tee-square and set-squares.—The tee-square is generally made of pear wood or mahogany. Its blade should be of length suited to the drawing-board, and should taper, being wider near the haft, a form designed to combine rigidity with lightness and balance. The blade is fixed to the upper surface of the haft, the latter thus offering no impediment to the sliding of the set-squares. The working edges of the haft and blade should be truly straight; it is not absolutely necessary, though desirable, that they should be at right angles. The haft of the tee-square slides along the left edge of the board, allowing parallel lines to be drawn along the blade horizontally on the paper.

Two **set-squares** are necessary, with angles respectively of 60° and 45°. The former may be from 8″ to 10″ long, and the latter 5″ to 6″. The thickness should not be more than $\frac{1}{10}''$, and bevelled edges should be avoided since it is necessary to be able to use the set-squares either face up. Unless made of homogeneous material like vulcanite or



Test of a Straight-Edge.



Test of the Right Angle of a Set-Square.

celluloid, they should be "framed" to prevent distortion by shrinkage of the wood across the grain.

To **test** and **adjust** the **set-squares**. The edges can be made straight by using the trying plane, but it is further necessary that the angles be accurate.

The right angle must first be tested and set, as follows:—Place the set-square ABC with the edge BC on a straight edge TT′. Draw a line on the paper along the edge CA. Now turn the square over into the position A′B′C, and again draw a line along the edge CA′. These two lines ought to coincide. If they do not, recourse must be had to the trying plane.

The other angles must now be made true. For the 45° square the angles A and B should be equal; and for the 60° and 30° square one angle must be double the other. These conditions are easily tested by drawing lines on paper. If necessary the sloping edge AB must be adjusted with the plane until the angles are correct, several trials generally being required before the result is satisfactory.

7. The Clinograph.—This is a set-square which has a swivelling blade held in position by a friction joint. In conjunction with a tee-square it is used in drawing lines which are parallel or perpendicular to one another, or which are symmetrically inclined to horizontal or vertical lines.

Thus, having set the blade so as to lie along any given line PP, the clinograph may be brought to a position suitable for drawing any parallel line QQ by sliding the instrument along the tee-square and the latter along the drawing-board.

In drawing any line perpendicular to PP, such as RR, the edge of the clinograph in contact with the tee-square is changed.

When drawing a line such as SS, which is symmetrically inclined to RR, the instrument is turned over on to its other face, the same edge remaining in contact with the square.

For lines like PP, SS, which are required to make equal angles, one to the horizontal and the other to the vertical, both face and edge are changed.

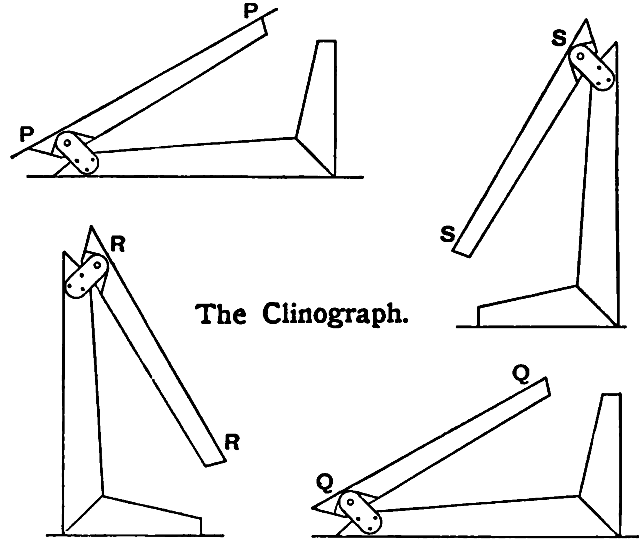
The clinograph should be tested for accuracy like the set-squares, and the edges trued up with the trying plane if necessary. All three edges should be straight, and two of them at right angles.

8. **The compasses and dividers.**—These vary much in quality and design, but very serviceable instruments suitable for use in this subject can be obtained at a moderate cost. Needle point fittings are not necessary, but "knee joints" are an advantage, as they permit of the compass legs being kept upright for circles of different sizes.

The compass lead should be of hard quality, say HHH, and sharpened to a narrow chisel edge with glass paper as previously described. The pencil point should be about $\frac{1}{32}''$ short of the steel point of the other leg. In case the latter gets damaged, it may be remedied on a fine oilstone.

Now let the student sharpen his pencils and work carefully the following exercises:—

9. Exercises for practice with the drawing-pencil and compasses, and for testing the squares.



The Clinograph.

1. Take the pencil compasses, and with the same centre O describe two circular arcs, one of $\frac{1}{4}''$ radius, the other of $2\frac{1}{4}''$ radius. On the latter arc step off with the dividers twelve points each $\frac{1}{10}''$ apart. From these points draw radii towards the centre O, stopping at the smaller arc. These lines are $\frac{1}{100}''$ apart at their inner ends, and should be so fine as to be each distinct.
2. Use the tee-square and the 45° set-square. With centre O, any radius, say 2″, describe a circle. Draw the horizontal and vertical tangents to this circle, forming a square. Test whether the two lines drawn at 45° through O, and sloping opposite ways, pass through the corners of the square.
3. Use the tee-square and the 60° set-square. Draw a circle radius, say 2″. Draw six lines touching the circle and forming a regular hexagon. Prick off the six corners. Apply the dividers to test whether the six sides are all exactly equal in length. Try whether lines can be drawn along the edges of the squares so as to pass through the opposite corners.

10. Scales. The Protractor.—In Practical Geometry scales are in constant use for measuring lines, and for marking off lengths to numerical data.

Fig. (*a*) is a scale the unit length of which is 1 inch. The unit is subdivided into 10, and each tenth into 2, equal parts. This may be called a **1 inch decimal scale.**

Fig. (*b*) shows a **decimal scale** of centimetres, the smaller divisions being millimetres. It might have been figured so as to read from both ends.

In Fig. (*c*) two **decimal scales** of $\frac{1}{8}$ inch and $\frac{1}{4}$ inch are combined.

Two types of scales are illustrated in these figures, known technically as "close divided" and "open divided" scales. In the former the subdivisions extend over the whole length, the zero point being at the end. See (*b*). In the latter, only one main division is subdivided, at the inner end of which is placed the zero. The close divided scale is perhaps a little more convenient to use.

The closest lines on a scale should not be more than about $\frac{1}{10}''$, nor less than about $\frac{1}{50}''$ apart. Thus in decimal scales the unit may be subdivided into 2, 5, 10, 20 or 50 parts according to its length. If the student possess a slide rule, he will see a good illustration of this.

In using a scale the dividers should not be employed. The scale should be **applied direct to the paper** and the required dimension **marked off with the pencil** (or pricker), or read off, as the case may be. Thus all divisions should extend to the edge of a scale.

The student will require decimal scales of $1''$, $\frac{1}{2}''$, $\frac{1}{4}''$, $\frac{1}{8}''$ and 1 cm.; also two other $1''$ scales subdivided into 8ths and 12ths. These may be bought printed on strips of thin cardboard. Or at a little extra cost the whole set of seven may be obtained on one "engine divided" boxwood scale, a suitable length for which is about 8 inches.

The **protractor** is a scale for measuring angles. We describe it in the next chapter.

Now let the student acquire facility in the use of scales by working such examples as the following :—

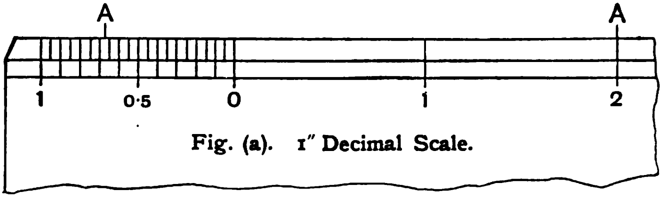


Fig. (a). 1″ Decimal Scale.

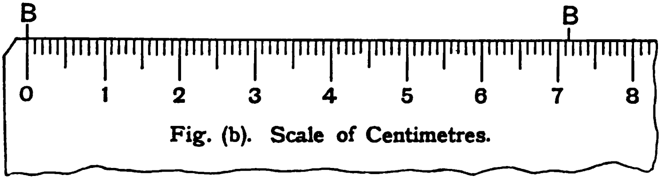


Fig. (b). Scale of Centimetres.

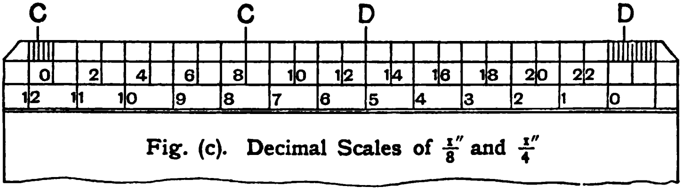


Fig. (c). Decimal Scales of $\frac{1}{8}''$ and $\frac{1}{4}''$

11. Examples on the use of scales.

1. Read off the lengths of AA and BB. *Ans.* 2.67″; 7.14 cm.
2. How many units does CC measure on the $\frac{1}{8}''$ scale? *Ans.* 8.68.
   If each unit represents 1 foot, what length does CC represent? *Ans.* 8.68 feet.
3. If D, D were two places on a map plotted to scale of $\frac{1}{4}''$ to 1 mile, what would be their distance apart? *Ans.* 5.32 miles.
   If the scale of the map were $\frac{1}{4}''$ to 10 miles, what would then be the distance DD? *Ans.* 53.2 miles.
4. Draw a line. Mark a point E on it. Apply scales to the line and mark off from E lengths of 2.6″, 3.18″, 5.33 cm.
5. Use the $\frac{1}{2}''$ scale to mark off (*a*) a length of 374 yards to the scale of $\frac{1}{2}''$ to 100 yards; (*b*) a length of 2.36 inches to the scale of $\frac{1}{2}''$ to 1″.

12. Standard of accuracy in graphical work.—Graphical constructions are always more or less inaccurate owing to faulty instruments, want of practice in their use, imperfections of the human faculties, etc. Recognising this, and remembering that his work will be largely quantitative, it will be well for the student to acquire some knowledge as to the standard that is practically possible under ordinary working conditions. Let some actual tests be made by students themselves. We suggest experiments like the following; they should be carried out by groups or by a class of students for their mutual information.

Thus, can a given length be estimated or marked off correctly to within one millimetre? To test this, experiments 1 and 2 may be made.

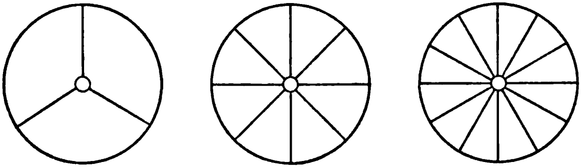
Experiment 1.—Let two points be marked on a line, and let their distance apart be measured independently on a centimetre scale by members of the group, and the results compared.

Experiment 2.—Let each student select his own length, mark it off, then pass it to his neighbour to measure. Let the discrepancies in the various pairs of measurements be made known.

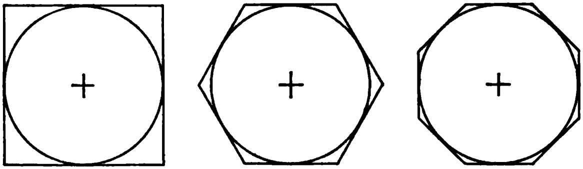
Experiment 3.—Let a circle of specified radius be drawn by each student. Then compare the actual sizes of the circles by passing a strip of paper round the group, from a point on the edge of which the several diameters are to be marked off.

Experiment 4.—Mark a point O near one corner of the paper. Use the tee and set-squares to draw two perpendicular lines through O. On these lines scale off given lengths OA, OB, say 20.5″ and 14.3″, the longer the better. Prick off A and B. Compare the distance between A and B in each case by means of a paper strip passed round the class as in Experiment 3. Note the discrepancies. Calculate the true distance AB. Complete the rectangle on OA, OB, by using the tee and set-squares. Compare the lengths of the two diagonals, recording the difference.

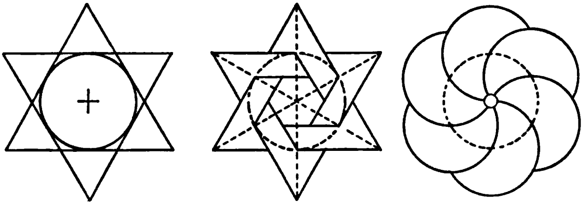
Ex. 1.



Ex. 2.



Ex. 3.



13. Examples illustrating the use of drawing instruments.

***1.** Draw six circles each of 1″ radius, their centres 3″ apart. Then draw a series of 3, 4, 6, 8, 12, 24 equidistant radii, as shown above for the cases of 3, 8, and 12.

Note 1.—In the last case the 60° and 45° set-squares must be used in conjunction.

Note 2.—The student should remember for future use this expeditious manner of dividing the circumference of any circle into 2, 3, 4, 6, 8, 12, or 24 equal parts.

***2.** Draw six circles as in Ex. 1. Then, as partially indicated in the second row of figures draw lines touching these circles so as to form a series of regular polygons of 3, 4, 6, 8, 12 and 24 sides.

***3.** Draw three figures like those given, but enlarged, so that in each case the circle is 1″ diameter.

***4.** Describe a circle of $\frac{3}{4}$″ radius. Then carefully draw, in succession, the series of circumscribing squares, so as to obtain a figure similar to the one shown. Then measure accurately, and write down the lengths of the four sides A B, and the two diagonals A A, B B. *Ans.* 4.64", 6.56".

N.B.—All lines may be drawn by the use of the tee-square and the 45° and 60° set-squares, without the employment of a protractor. (1901)

***5.** Use the tee-square and the clinograph. First draw a circle of $\frac{1}{2}$" diameter. Then, with the blade of the clinograph set at a small angle, say about 10°, draw eight lines touching the circle so as to form the figure shown. Prick off the four corners, and test whether these points are all equidistant from the centre.

***6.** Interlacing pattern for border founded on squares. First draw the two outside perpendicular lines. Then apply the scale and mark off on both lines the seven sides, each 0.2". Draw the squares faintly, and line in the pattern.

***7.** Pattern based on squares and composed of semicircles.

***8.** Patterns made up of circular arcs and founded on equilateral triangles.

14. Definitions. Summary of Chapter I.—A **point** is conceived as having position in space, but not magnitude.

Note.—On a drawing a point must be of visible size. It may be shown as the intersection of two lines; or by a very small circle; or by a hole pricked in the paper. See the figure, p. 14.

A **line** is conceived as having length, but not breadth nor thickness. A line may be straight, curved, or zig-zag. See the figure, p. 14.

A **straight line** is one that would not enclose a space if it were rotated about any two points in it.

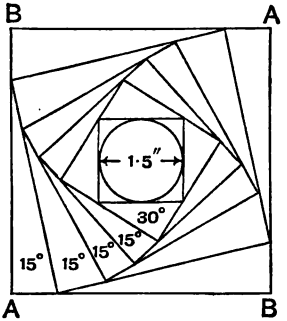
Notes.—A line drawn on paper must have a visible width; but by using a properly sharpened pencil this is very small.

A line of finite extent should have its ends defined by short cross lines, or by small circles with their centres at the ends.

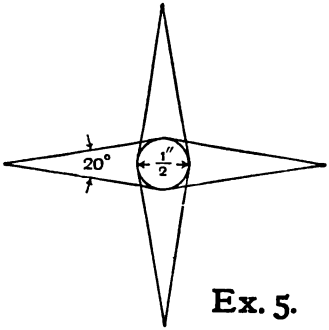
The property of a straight line stated above is made use of in testing the truth of a straight edge. See Art. 5.

When a "line" is spoken of without qualification, a *straight* line is generally to be understood.

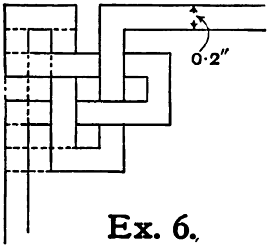
A **surface** is conceived as having length and breadth, but not thickness.



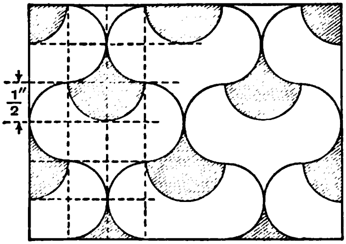
Ex. 4.



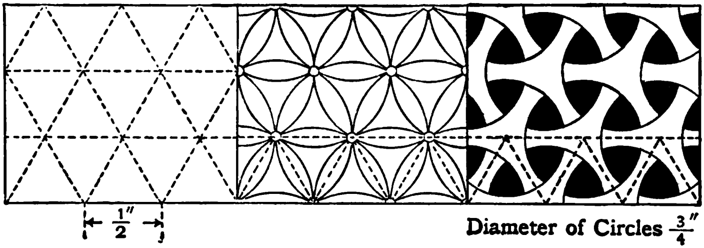
Ex. 5.



Ex. 6.

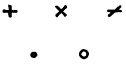


Ex. 7.

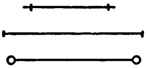


Diameter of Circles $\frac{3}{4}''$

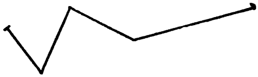
Ex. 8.



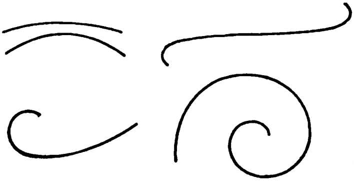
Points.



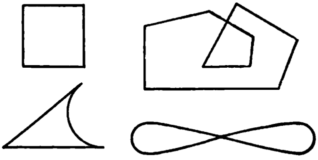
Lines of Definite Length.



Zig-Zag Line.



Curves.



Plane Figures.

A **plane** is a flat surface; a straight line containing any two points in the plane lies wholly in the plane.

A **plane figure** is a portion of a plane bounded or defined by curved or straight lines.

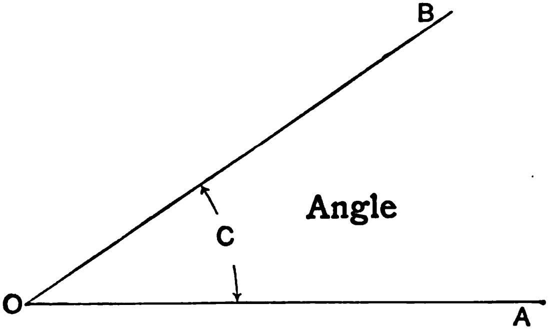
Note.—The word *figure* is by custom frequently used to denote the boundary lines, as well as the enclosed space. Thus the circumference of a circle is often referred to as the circle; we shall have to use the word in this sense. In any particular case the meaning is to be inferred from the context.

Should the student not understand the meaning of any geometrical term used in the text, let him refer to the index at the end of the volume.

CHAPTER II

THE MEASUREMENT OF ANGLES

15. An **angle** is formed when two straight lines meet



Thus in the above figure, O A and O B define an angle. O is called its **apex** or **corner.** O A, O B are its **arms** or **bounding lines.** The angle may be referred to by naming its apex as O; or by indicating its arms, as A O B, or B O A, the apex letter being in the middle; or by labelling the angular space, as C.

The angle may be conceived as having been generated by the line O B turning from the position O A about O into its present position.

16. The magnitude of an angle.—Any angle possesses magnitude. Its size, however, cannot be estimated as a length, or an area; it is different in kind to these.

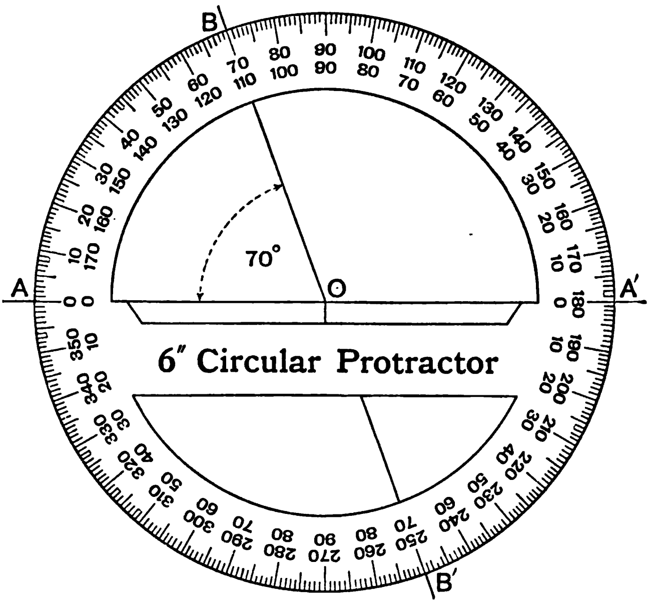
If the student were to cut out an angle in paper, and fold it so as to bring its arms together, the line of fold would divide the angle into two equal angles, each half the size of the original. Similarly he could make angles of one-third the size, or of double the size, etc. Thus the magnitude of an angle is measured by, and expressed in terms of some angle of known and specified size, called the unit angle. There are various units in common use, all angles, just as there are various units for measuring, say, a length, as the inch, foot, metre, mile, etc., all of which are lengths.

Our best standpoint is that which, as previously stated, regards any angle as having been generated by the rotation of one of its arms. The size of the angle would thus be stated as the *amount of rotation* required to form it. The unit might be one complete turn or revolution; with this measure a right angle would be expressed as an angle of $\frac{1}{4}$ turn. Mathematicians think of angles of many turns; and even of negative angles when the arm rotates backwards.

17. The measurement of angles in degrees. The protractor.—The usual plan, however, is to adopt as the unit angle not a whole revolution, but the $\frac{1}{360}$th part of a revolution. This is called an angle of one degree, or a degree, written 1°. One complete turn thus contains 360°. The angles of the set-squares are 90°, 60°, 45°, and 30°, *i.e.* angles of $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, and $\frac{1}{12}$ of a turn respectively; hence their general utility in graphical work.

The degree is subdivided into 60 parts, called minutes, and each minute into 60 parts, called seconds. These angles are, however, too minute to be dealt with in our work. We shall find it convenient to express the subdivisions of a degree in decimals, and we cannot measure beyond the first decimal.

A **circular protractor**, for plotting and measuring angles in degrees, is illustrated. After what has been said, the manner of using it should be self-evident. It is repre-



sented in the figure as it would be employed when measuring the angle AOB, or when setting off from O, in OA, a line OB making 70° with OA. To ensure maximum accuracy, a diameter of the instrument, and not merely a radius, should be utilised; that is, AO should be extended to A′, and B′ marked as well as B.

The cost of this circular protractor would usually be prohibitive unless made of cardboard. A 6″ semicircular one, or a 6″ rectangular boxwood protractor, would be very serviceable, if carefully divided. It would be a capital exercise for a student to make his own protractor.

18. A practical method of measuring the length of any plotted curve.—We shall require this construction before proceeding further. It is best explained by taking an actual example. Thus, suppose we wish to measure on a map the length of a stretch RR of a river. See the figure.

On a piece of firm tracing paper rule a straight line. Mark a point **r** on this line. Bring **r** over one end R, and place the sharp point of a pencil or a pricker at **r**. Turn the tracing paper until line and river also coincide at a second point near **r**. Shift the pricker to this point and rotate slightly, bringing a third adjacent point into coincidence. Continue step by step until the other end R is reached, over which mark a second point **r**. Then **rr** gives the length of RR.

After a few trials this process will be found very expeditious, and it is capable of great accuracy.

By a similar operation we could mark off on a curve a length **rr**.

Examples 1.—If the scale of the map be 1 inch to 10 miles, what is the length of RR? *Ans.* 28.3 miles.

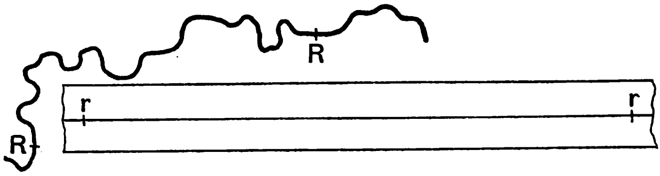
2. Draw any circle. Find the length of its circumference as above. Also measure the diameter. Divide the circumference by the diameter. *Ans.* 3.142 or $3\frac{1}{7}$.

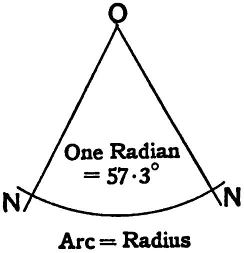
19. The measurement of angles in radians.—Another unit of measurement is the **radian**, an angle of 57.3°. The student will think this a curious unit; yet it is a natural and very important one. A radian is an angle such that arc = radius. See the figure. Let the student make the following experimental measurements:—

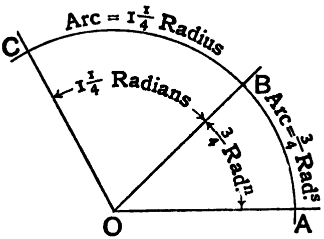
First. With centre O and any radius describe an arc. Lay off the radius on tracing paper, and use this as just explained, to mark off a length NN of arc equal to the radius. Draw the radii ON, ON. Measure the angle NON in degrees by means of a protractor. *Ans.* 57.3°. The angle NON is one radian. Let this be repeated with different radii until the student convince himself that in all cases when arc = radius, then angle = 57.3°.

Next. Construct a figure like the one shown, selecting any radius, and making arc AB = $\frac{3}{4}$ radius, and arc BC = $1\frac{1}{4}$ radius, and therefore arc AC = $\frac{3}{4}$ radius + $1\frac{1}{4}$ radius = 2 radius. Use the protractor and confirm that angle AOB = 43° = $\frac{3}{4}$ radian, angle BOC = 81.6° = $1\frac{1}{4}$ radians, and angle AOC = 114°.6 = 2 radians.

Third. Draw any angle at random; measure it in radians as follows; with the apex as centre describe an arc







subtending the angle; measure the length of the arc by the use of tracing-paper; divide the length by the radius. The answer gives the magnitude of the angle in radians. Multiply by 57.3 to convert to degrees. Compare this with the angle as measured by the protractor.

The student should experiment for himself in this manner until he is convinced that arc ÷ radius is a measure of the magnitude of any angle, giving its value in radians; one radian being equal to 57.3°. He will soon find the advantage of taking the radius 10 units long on a convenient scale, say on the ¼″ or on the centimetre scale, and measuring the arc on the same scale.

Examples.—1. Find by construction the number of radians in a right angle. *Ans.* 1.571.

2. If the radius of a circle is unity, what is the length of the semicircumference? *Ans.* 3.142 units. How many radians are there in 180°? *Ans.* 3.142.

20. The Mariners' Compass.—This illustrates yet another way of reckoning angles, which a sailor estimates by **points.** As will be seen from our illustration there are 32 points round the dial of the instrument; thus one point corresponds to $\frac{1}{32}$ of a revolution, or to $\frac{1}{8}$ of a right angle, that is, to $11\frac{1}{4}°$. We shall have occasion later to make use of this method in giving the data to some of the problems. The names by which the points of the compass are known should be evident from the figure. Thus N*b*E (read North by East) signifies a direction of $11\frac{1}{4}°$ East of North. And E25°N means 25° N of E.

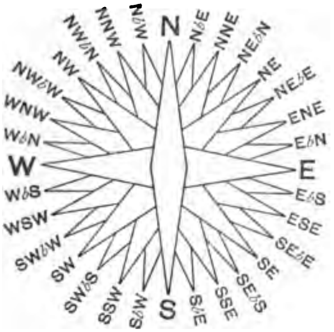
21. Chords of angles.—Let the student draw any angle on his paper. Then by the use of compasses, squares, and scale (but not a protractor), let him obtain some information by construction and measurement, which, being supplied to his neighbour, would enable the latter to reproduce the angle. He is not asked to measure the magnitude of the angle; only to supply data which would enable the angle to be drawn.

There are numerous ways of doing this. Thus, mark two points A and B on the arms, and, calling the apex O, measure and give the lengths of OA, OB, and AB. This is a sufficient yet cumbrous method involving *three* lengths; let us seek a simpler construction.

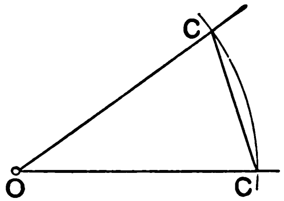
In this article and the succeeding one we show four ways of solving this problem. The student should study these with care. He is making the acquaintance of constructions and numbers of great importance relating to angles.

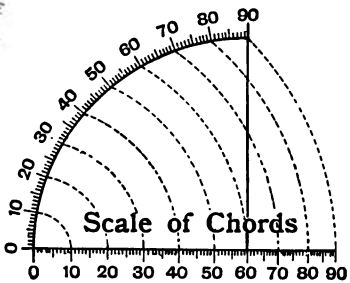
With the apex O of the angle as centre, any radius, describe the arc CC. Draw the *chord* CC of this arc. Then the angle could be constructed if we knew the lengths of the chord and radius. But we can avoid giving both lengths. It would be sufficient if we gave simply the number obtained by dividing the chord by the radius. The ratio **chord ÷ radius** is called the **chord of the angle.**

Thus if the student apply a $\frac{1}{8}''$ scale to the figure, he will find that the radius OC measures 10 units, and the chord CC 6.1 units. The chord of the angle is therefore 6.1 ÷ 10, that is 0.61.



Mariners Compass.





Let the student draw an angle and describe arcs of different radii, measuring the corresponding chords, and satisfy himself that for the **same angle the ratio chord ÷ radius is the same number, whatever the radius may be.**

A **table of chords** of angles is given on p. 25. This can be used either for plotting an angle given in degrees, or for measuring the number of degrees in a given angle. The radius of the circle should be taken equal to unity, or 10 read as unity, on any convenient scale.

A **scale of chords** may be used instead of a table. Its construction and manner of use should be evident from the figure. The radius of the circle is the length from 0 to 60 on the scale.

22. The sine, cosine, and tangent of an angle.—We now give the three remaining constructions promised in the last article.

Let an angle AOB be drawn. Mark any point P in one arm and draw a perpendicular PM to the other arm.

We thus have a right-angled triangle formed; for distinction let us call the three sides, the *perpendicular*, *base*, and *hypotenuse* as in the figure.

Now the student will see at once that if any two sides of this triangle were measured and stated, the triangle could be constructed from this data. He should further understand that if the *ratio* of any pair of sides were given, that is, the number obtained by dividing one side by the other, the angle could be plotted. For example, he could set out the angle if he were told that the perpendicular was half the base; or in other words, if the ratio of perpendicular to base was 1 to 2 or 1 ÷ 2, or $\frac{1}{2}$, or 0.5. He would simply mark off O M of any length, say 10 units on a convenient scale; draw the perpendicular M P of half the length, that is 5 units on the same scale; and join O P.

These ratios are so important that special names have been given to them. Let the student commit these to memory.

$\frac{\mathbf{PM}}{\mathbf{OP}}$ or $\frac{\textbf{perpendicular}}{\textbf{hypotenuse}}$ is called the **sine** of the angle AOB.

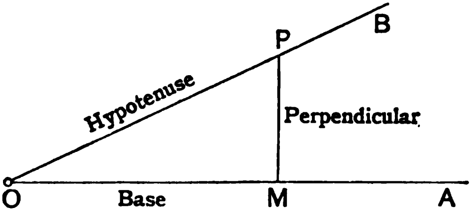
$\frac{\mathbf{OM}}{\mathbf{OP}}$ or $\frac{\textbf{base}}{\textbf{hypotenuse}}$ is called the **cosine** of the angle A O B.

$\frac{\mathbf{PM}}{\mathbf{OM}}$ or $\frac{\textbf{perpendicular}}{\textbf{base}}$ is called the **tangent** of the angle AO B.

These are generally written in the abbreviated forms, **sin AOB**, **cos AOB**, and **tan AOB**. But in speaking let the full names be pronounced.

Tables of the value of these functions are given on p. 24, the manner of reading the tables being there illustrated by examples.

As in the previous constructions of Arts. 19 and 21, for radians and chords, let the student verify by actual experimental measurements, that for the **same angle** these ratios for sine, cosine, and tangent are **independent of the**



actual size of the triangle OPM; that is P may be taken **anywhere on either arm.**

23. Experiments for ascertaining the degree of accuracy attainable in the measurement of an angle.—The following tests may be made. Let all the students in the class be supplied with copies of the same angle, which may be printed, or carefully drawn by using a prepared templet of sheet material or otherwise produced. The length of the arms should not be less than 6″. Then make the following measurements, and enter up the answers systematically in tabular form, for comparison with one another, and with the average values.

1. Measure the angle with the protractor.
2. Measure the angle in radians by the use of tracing paper. Convert into degrees by multiplying by 57.3.
3. Measure the angle by using a scale of chords.
4. Measure the angle by the use of a table of chords.
5. Measure the sine of the angle. Then use the table of sines and read off the angle in degrees.
6. Measure the cosine of the angle. Use the table of cosines to find the number of degrees.
7. Measure the tangent of the angle. From the table of tangents deduce the angle in degrees.

24. Trigonometrical tables.—On the next two pages we give tables of the values of the sines, cosines, tangents, and chords of angles from 0° to 90°, increasing by intervals of 0.1°. We also give a table of degrees and radians. The method of taking out values from the tables is illustrated by three examples.

Trigonometrical Tables.

SINES OF ANGLES

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	Difference to be added.								
											·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.122	.139	.156	2	3	5	7	9	10	12	14	16
10°	.174	.191	.208	.225	.242	.259	.276	.292	.309	.326	2	3	5	7	8	10	12	13	15
20°	.342	.358	.375	.391	.407	.423	.438	.454	.469	.485	2	3	5	6	8	9	11	12	14
30°	.500	.515	.530	.545	.559	.574	.588	.602	.616	.629	1	3	4	6	7	9	10	11	13
40°	.643	.656	.669	.682	.695	.707	.719	.731	.743	.755	1	2	4	5	6	7	9	10	11
50°	.766	.777	.788	.799	.809	.819	.829	.839	.848	.857	1	2	3	4	5	6	7	8	9
60°	.866	.875	.883	.891	.899	.906	.913	.920	.927	.934	1	1	2	3	4	4	5	6	7
70°	.940	.946	.951	.956	.961	.966	.970	.974	.978	.982	0	1	1	2	2	3	3	4	4
80°	.985	.988	.990	.993	.995	.996	.998	.999	.999	1.000	0	0	0	1	1	1	1	1	1

COSINES OF ANGLES

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	Difference to be subtracted.								
											·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	1.000	1.000	.999	.999	.998	.996	.995	.993	.990	.988	0	0	0	1	1	1	1	1	1
10°	.985	.982	.978	.974	.970	.966	.961	.956	.951	.946	0	1	1	2	2	3	3	4	4
20°	.940	.934	.927	.920	.913	.906	.899	.891	.883	.875	1	1	2	3	4	4	5	6	7
30°	.866	.857	.848	.839	.829	.819	.809	.799	.788	.777	1	2	3	4	5	6	7	8	9
40°	.766	.755	.743	.731	.719	.707	.695	.682	.669	.656	1	2	4	5	6	7	9	10	11
50°	.643	.629	.616	.602	.588	.574	.559	.545	.530	.515	1	3	4	6	7	9	10	11	13
60°	.500	.485	.469	.454	.438	.423	.407	.391	.375	.358	2	3	5	6	8	9	11	12	14
70°	.342	.326	.309	.292	.276	.259	.242	.225	.208	.191	2	3	5	7	8	10	12	13	15
80°	.174	.156	.139	.122	.105	.087	.070	.052	.035	.017	2	3	5	7	9	10	12	14	16

TANGENTS OF ANGLES

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	Difference to be added.								
											·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.123	.141	.158	2	4	5	7	9	10	12	14	16
10°	.176	.194	.213	.231	.249	.268	.287	.306	.325	.344	2	4	6	7	9	11	13	15	17
20°	.364	.384	.404	.424	.445	.466	.488	.510	.532	.554	2	4	6	8	11	13	15	17	19
30°	.577	.601	.625	.649	.675	.700	.727	.754	.781	.810	3	5	8	10	13	16	18	21	23
40°	.839	.869	.900	.933	.966	1.000	1.036	1.072	1.111	1.150	4	7	11	14	18	21	25	28	32
50°	1.19	1.23	1.28	1.33	1.38	1.43	1.48	1.54	1.60	1.66	1	1	2	2	3	3	4	4	5
60°	1.73	1.80	1.88	1.96	2.05	2.14	2.25	2.36	2.48	2.61									
70°	2.75	2.90	3.08	3.27	3.49	3.73	4.01	4.33	4.70	5.14									
80°	5.67	6.31	7.12	8.14	9.51	11.43	14.30	19.08	28.64	57.29									

CHORDS OF ANGLES

											Difference to be added.								
	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	·1°	·2°	·3°	·4°	·5°	·6°	·7°	·8°	·9°
0°	.000	.017	.035	.052	.070	.087	.105	.122	.140	.157	2	3	5	7	9	10	12	14	16
10°	.174	.192	.209	.226	.244	.261	.278	.296	.313	.330	2	3	5	7	9	10	12	14	16
20°	.347	.364	.382	.399	.416	.433	.450	.467	.484	.501	2	3	5	7	9	10	12	14	15
30°	.518	.534	.551	.568	.585	.601	.618	.635	.651	.668	2	3	5	7	8	10	12	13	15
40°	.684	.700	.717	.733	.749	.765	.781	.797	.813	.829	2	3	5	6	8	10	11	13	14
50°	.845	.861	.877	.892	.908	.923	.939	.954	.970	.985	2	3	5	6	8	9	11	12	14
60°	1.000	1.015	1.030	1.045	1.060	1.075	1.089	1.104	1.118	1.133	1	3	4	6	7	9	10	12	13
70°	1.147	1.161	1.176	1.190	1.204	1.218	1.231	1.245	1.259	1.272	1	3	4	6	7	8	10	11	12
80°	1.286	1.299	1.312	1.325	1.338	1.351	1.364	1.377	1.389	1.402	1	3	4	5	6	8	9	10	12

DEGREES AND RADIANS

	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	Diff. to be added.	
0°	.000	.017	.035	.052	.070	.087	.105	.122	.140	.157	0.1°	.002
10°	.174	.192	.209	.227	.244	.262	.279	.297	.314	.332	0.2°	.004
20°	.349	.366	.384	.401	.419	.436	.454	.471	.489	.506	0.3°	.005
30°	.524	.541	.558	.576	.593	.611	.628	.646	.663	.681	0.4°	.007
40°	.698	.716	.733	.750	.768	.785	.803	.820	.838	.855	0.5°	.009
50°	.873	.890	.908	.925	.942	.960	.977	.995	1.012	1.030	0.6°	.010
60°	1.047	1.065	1.082	1.100	1.117	1.134	1.152	1.169	1.187	1.204	0.7°	.012
70°	1.222	1.239	1.257	1.274	1.291	1.309	1.326	1.344	1.361	1.379	0.8°	.014
80°	1 395	1.414	1.431	1.448	1.466	1.483	1.501	1.518	1.536	1.553	0.9°	.016

Examples.—1. Required the sine of 34·5°. Referring to the table of sines, we find

Sine of 34° . ·559
Add for .5° diff. . 7
Sine of 34.5° . .566

2. Required the angle whose chord is ·824.

From the table, the chord of 48° is ·813
For the difference 11 add .7° 11
The required angle is 48.7°

3. Required the cosine of 41·4°.

Cosine of 41° . . ·755
For excess .4° subtract diff. 5
Cosine of 41.4° . . .750

25. Some definitions relating to angles.—We now explain and illustrate by diagrams some additional terms in common use.

An **acute angle** is an angle less than 90°.

An **obtuse angle** is greater than 90° and less than 180°.

An **angle greater than 180°** may be indicated as in the figure, where an angle of 300° is shown. The minute hand of a clock would describe an angle of this magnitude in 50 minutes, and the hour hand in 10 hours.

Note.—In the above definitions negative angles are not considered.

Angles are said to be **complementary** when their sum is 90°. One angle is the **complement** of the other.

Angles are said to be **supplementary** when their sum is 180°. One angle is the **supplement** of the other.

Parallel lines may be regarded as lines the angle between which is **zero.**

26. Miscellaneous examples and experiments.

1. A person facing N.N.E. turns part way round so as to face W.b.S. Through what angle does he move if the direction of turning is (*a*) towards the left, (*b*) towards the right? *Ans.* $123\frac{3}{4}^\circ$, $236\frac{1}{4}^\circ$.

2. Construct an angle whose sine is 0.3. Measure its chord. *Ans.* 0.305.

3. Construct an angle whose cosine is 0.4. Measure its sine. *Ans.* 0.916.

4. Construct an angle whose tangent is 0.7. Measure the angle in degrees. *Ans.* 35°.

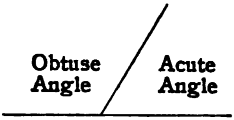
5. Draw any angle A, and verify by construction, measurement, and numerical calculation that

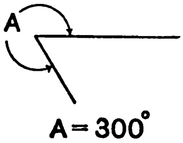
$$(a)\ \tan A = \frac{\sin A}{\cos A}. \qquad (b)\ (\sin A)^2 + (\cos A)^2 = 1.$$

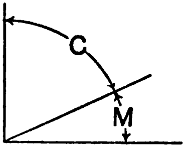
6. Draw any two complementary angles, and verify by construction and measurement that the sine of one is equal to the cosine of the other. Or in symbols, that

$$\sin C = \cos(90 - C), \text{ or } \cos C = \sin(90 - C).$$

7. Select two adjacent walls in a room or elsewhere, and by the use of a tape-measure and table of chords find the angle

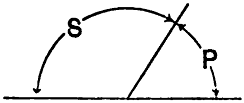






Complementary Angles

$C + M = 90^\circ$
$C = 90^\circ - M$
$M = 90^\circ - C$



Supplementary Angles

$S + P = 180^\circ$
$S = 180^\circ - P$
$P = 180^\circ - S$

between the walls, or confirm that this angle is 90° if the room is rectangular.

8. Secure one end of a piece of string to the floor at one corner of a room. Then stretch the string along the floor, and with the assistance of a tape-measure and table of chords adjust its position so that it shall make a given angle, say 50°, with the wall.

9. Let a straight line several yards in length be marked on the floor of a room as follows :—Take a piece of string, rub it well with chalk, stretch it tightly on the floor, lift up the middle of the string vertically and then let go. Now mark a point A in this line, and with the aid of a tape-measure, table of chords, and chalked string, mark a second line passing through A and making a given angle, say 70° or 90°, with the first line.

10. If opportunity permit, assist in setting out a cricket crease, a tennis lawn, or a football field.

11. Let the student try and devise some simple piece of apparatus by means of which he can estimate approximately the angle subtended by two distant objects, such as two trees.

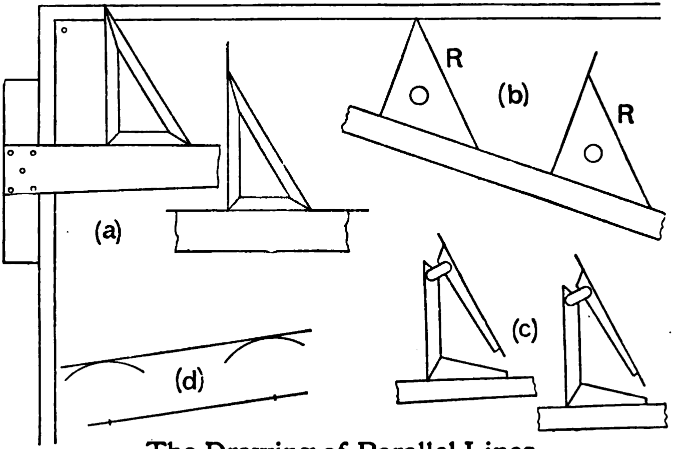
CHAPTER III

SOME FUNDAMENTAL CONSTRUCTIONS

27. Introductory.—The constructions with which we are concerned in this chapter recur so frequently in all our graphical work that it is very desirable that the student shall make a preliminary study of the various ways of effecting them with accuracy and dispatch. At the same time the constructions themselves may be utilised to introduce and illustrate important elementary geometrical truths. We refer to processes like the drawing of parallel and perpendicular lines, circles, the bisection of lines, angles, etc. The fundamental character of these constructions will be appreciated if they are compared with the analogous operations of the machine tools in an engineer's workshop in the production of parallel and perpendicular plane surfaces and of "turned" surfaces, and which constitute the bulk of the workshop processes. To ensure success the draughtsman, like the workman, must know how to handle his tools, and must keep them in good order.

28. The drawing of parallel lines.—Our illustrations show the common methods employed.

(a) **Horizontal and vertical parallel lines.—By use of tee and set-squares.**—This case scarcely needs illustration, except to indicate that the contact of the haft of the tee-square should be confined to the left edge of the board, and horizontal lines drawn from left to right. Also that it is the usual practice to place the set-square with its upright edge on the left, and to draw vertical lines upwards.



The Drawing of Parallel Lines.

(b) Parallel inclined lines. By use of straight edge and set-square.—The order of procedure should be:—First, lay the straight edge roughly parallel to the average direction RR between the lines. Then select the most suitable angle from the available ones of 30°, 45°, 60°, and 90° of the squares. Now make the final adjustment, slide the square, and draw the line. The manipulation is somewhat troublesome when the lines are any considerable distance apart, and nearly in the same line.

(c) Parallel inclined lines. By use of tee-square and clinograph.—This method gives a wider range than (*b*), and the result is attained with greater ease. It will facilitate the setting of the blade to remember that its turning causes little movement near the joint.

(d) By use of compasses and straight edge.—Set the compasses to the distance between the two lines; select two centres in the first line; describe the two arcs; draw the required line to touch these arcs.

29. The drawing of perpendicular lines.

(a) Horizontal and vertical lines.—These require merely the use of the tee and set-squares. See Fig. (*a*) p. 29.

(b) Inclined perpendicular lines. By use of straight edge and set-square.—Call the two perpendicular edges of the set-square P and Q, and the sloping edge S.

Edge S may rest on the straight edge, the latter being set so that P coincides with the given line; the perpendicular line is then drawn along Q.

Or P may rest on the straight edge with S lying along the first line. The set-square is then turned so as to have Q on the straight edge, that is through a right angle, and the second line is now drawn along S.

(c) The same. By use of tee-square and clinograph.—The blade is set with one of the perpendicular edges resting on the tee-square. Before drawing the perpendicular line the clinograph must be turned so that its other perpendicular edge rests on the tee-square, that is through an angle of 90°.

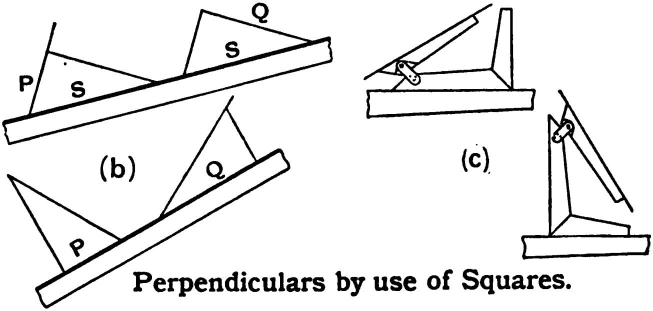
(d) By the aid of compasses.—The problem would usually present itself thus:—*Through a given point* P *to draw a perpendicular to a given line.* P may be in the line, or on one side.

Construction.—With centre P, any suitable radius, describe arcs cutting the line in A, A. With centres A, A, the same or another suitable radius, describe arcs intersecting in B. Then the line through PB is the required perpendicular.

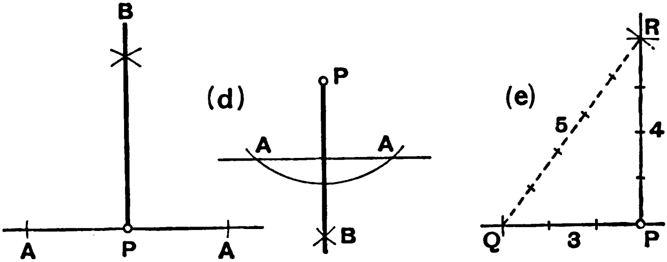
(e) By constructing a right-angled triangle.—This method is suitable for the case in which P is in the line near one end, and for some reason the line may not be produced.

Mark off PQ 3 units long on a suitable scale. With centres P and Q, radii 4 and 5 units, describe arcs intersecting in R. Draw PR the required perpendicular. See Art. 41 (*b*).

30. Geometrical theorems.—From the sliding of the



Perpendiculars by use of Squares.



Perpendiculars by aid of Compasses.

clinograph on the straight edge when drawing parallel lines, and its further rotation through 90° when drawing perpendicular lines, the student may learn some important lessons. Thus :—

(a) *If any figure move in a plane so that one line in it remains parallel to its original direction, then all the lines of the figure remain parallel to their original directions.*

(b) *If any figure be moved in its plane so that a line makes an angle A with its original direction, then all lines make the same angle A with their original directions.*

(c) *If two lines are respectively parallel or perpendicular to two other lines they contain the same angle or its supplement.*

31. The bisection of the line, arc, and angle.

(**a**) *The line* AA *is given.* With centres A, A and any suitable radius describe arcs intersecting in P, P. Then PP is the **perpendicular bisector** of AA.

(**b**) A usual method, applicable to either a straight line BB or circular arc BB, is to first take the radius equal to half the length as nearly as can be guessed, and describe arcs DD. Then with a second and more accurate guess describe two other arcs, between which the required mid point C can be located.

(**c**) *To bisect a given angle* O. With the apex O as centre, any suitable radius, cut the arms in E, E. With centres E, E, and the same or other suitable radius, describe arcs intersecting in F. Then OF is the required bisector.

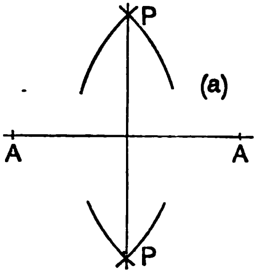
32. The division of a line into any number of equal parts.

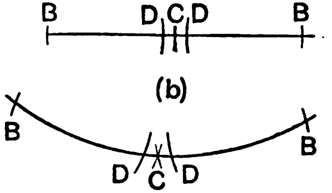
(**d**) *By continued bisection*, when the number is 2, 4, 8, 16, etc. In the figure GG is shown thus divided into 4 parts. If these were each bisected we should have 8 equal divisions, and so on.

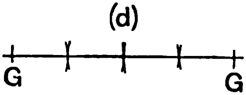
(**e**) *By aid of a scale.*—This is an excellent method and always applicable. To illustrate, let it be required to divide HH into, say, 7 equal parts.

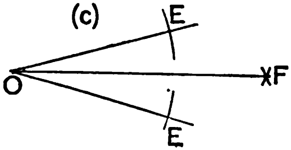
Through H, H draw any two parallel lines KK, KK, preferably at right angles to HH. Select a scale of convenient unit length, and adjust it as shown, so that 7 divisions are comprised between the two parallels. Mark the points of division along the edge with a pricker or fine conical pointed pencil. Remove the scale, and through these points draw lines parallel to KK, to meet HH in the required points of division.

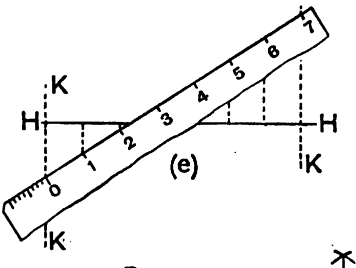
33. Examples and suggestions.—The student should endeavour to supplement the constructions of this chapter by others of his own invention. The use of tracing paper will often lead to interesting and useful devices. The following examples are intended to be mostly suggestive :—

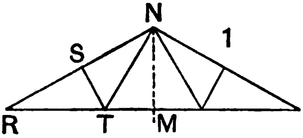


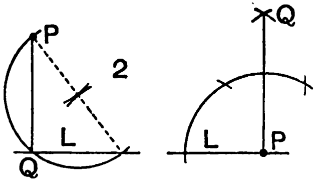












***1.** Draw the given figure to the scale of 1 inch to 10 feet, making RM = 18 feet, and MN = 11 feet long. Let ST bisect RN at right angles. Draw the perpendiculars and symmetrically-inclined lines by use of the clinograph. Measure the lengths of RT and TN. *Ans.* 12.37 feet.

***2.** P is a given point, L a given line. Verify the two methods shown of determining the perpendicular PQ. Invent two other methods.

3. Draw any angle and bisect it by the aid of a circle described on tracing paper.

4. Divide a line into a given number of equal parts by the use of tracing paper ruled with equidistant parallels.

5. Bisect a given line (*a*) by using two decimal scales, the unit of one being half that of the other; (*b*) by using one scale and arithmetical division.

6. Trisect a given line by using a scale and employing arithmetic.

7. Draw a circle of, say, 1½″ radius. Adjust the dividers by trial so that they shall step exactly five times round the circumference.

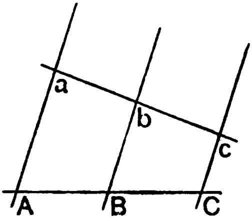
8. Trisect a given angle, it being permitted to divide an arc by trial with dividers or compasses into three equal parts.

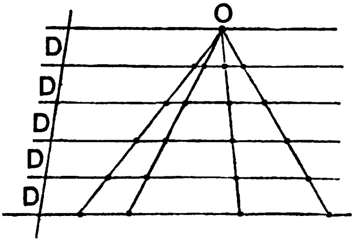
9. Mark a point on the floor of a room. By means of a measuring tape determine the foot of the perpendicular from the point to one of the walls. Verify the result by independent measurements.

CHAPTER IV

PARALLEL LINES. PROPORTIONALS

34. Properties of Equidistant Parallels.—On a straight line mark off two successive equal lengths, AB, BC. Let any three parallel straight lines be drawn through A, B, C. Now let these parallels be cut by any other straight line in a, b, c. Measure ab and bc; the lengths of ab and bc will be found to be equal. These parallel lines are equidistant; and it is thus seen that the intermediate parallel bisects any straight line terminated on the outer parallels.





Instead of *two* equidistant parallels we may have any number; and any straight line which cuts them is divided into segments which are all equal. Thus in the figure the segments D, D, D, D are all of equal length. And the student may verify by measuring the diagram, that all the straight lines radiating from O are divided into equal parts.

35. PROBLEM.—**To divide a given straight line into a given number of equal parts.**

The principles stated above may be applied in a variety of ways to the solution of this problem. We have already given one of the best of these in Art. 32 (*e*). The student would do well to invent for himself other ways of proceeding. We give one :—

Let it be required to divide L L *into five equal parts.*

From one end L draw a straight line in any convenient direction. On this line, starting from L, mark or step off five equal lengths. Join the end 5 of the last length to the other end L of the line. Through the other points 1, 2, 3, 4 draw parallels to this line 5 L. Then L L is divided as required into five equal parts.

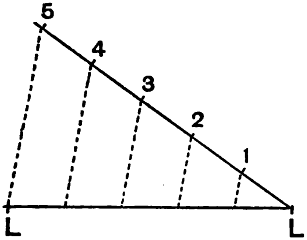
A good practical method is to use ruled tracing-paper. This is adjusted over the line, and the points of division pricked through. Or the line might be copied on tracing-paper and applied to ruled paper.

A plan frequently adopted, and which the student should practise, is to divide the line by trial with the dividers.

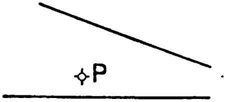
Sometimes it is convenient to measure the line with a scale, calculate the length of a division by arithmetic, and then set off these lengths with the scale.

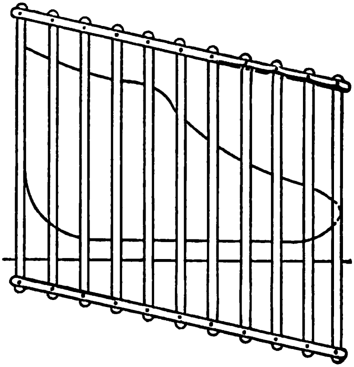
In connection with our present problem, we show a mechanical device, designed to facilitate the drawing of parallel lines at equal intervals apart, the amount of which may be varied. It is in the form of a series of jointed parallelograms, built up of strips of thin sheet steel. It is used principally by engineers when calculating the power of an engine, in order to divide the "indicator diagram" into ten parts by lines perpendicular to the atmospheric line. Our illustration shows it put to this use. For the properties of jointed parallelograms the student is referred to Ex. 2, Art. 57.

Examples.—1. Draw a straight line, and on it mark points A, B, C, D, 2″ apart as nearly as can be judged, using the



***3**





pencil only. Now apply the scale, and measure the lengths of AB, BC, CD, and AD, noting the differences from the intended lengths. Carefully bisect AB, BC, CD, employing a different method in each case. Find out which of the three methods has given the most accurate result.

2. Mark off two equal lengths, MC, CN, on a straight line. Take any point O, not in MN; join and produce OM, OC, ON. Now draw *any* straight line cutting these three radials in m, c, n. Examine whether mc is equal to cn. Find what condition or conditions must attach to the drawing of the line mcn in order that it shall always be bisected in c.

*3. Draw a figure something like that shown above, but larger. Through the point P draw a line to meet the two given lines in points equidistant from P.

4. Mark four lengths, each about 2″, as in Example 1. Divide these into seven equal parts, using a distinct method in each case. Say which method you prefer for accuracy, and which for dispatch.

5. Say how you would proceed if you had to divide a line several yards long, on the floor of a room, into a number of equal parts.

6. Describe an arc of 10″ radius, scale $\frac{1}{4}$, and draw two radii OA, OA, including an angle of 81°. Divide the arc AA into three equal parts (*a*) by trial with the dividers; (*b*) by setting off angles, each $\frac{1}{3}$ of 81°, employing the table of chords, p. 25.

7. Produce the radii of Ex. 6, and find out whether any transverse straight line can be drawn which shall be trisected by these radii.

36. Ratio and proportion.—By the **ratio** of two numbers we mean one divided by the other. Thus the ratio of 2 to 3 means $2 \div 3$, or $\frac{2}{3}$, or $2/3$; also often written $2 : 3$.

By **proportion** we signify the equality of ratios. Thus the ratio of 2 to 3 equals that of 4 to 6. In symbols

$$2 \div 3 = 4 \div 6, \text{ or } 2 : 3 = 4 : 6, \text{ or } \frac{2}{3} = \frac{4}{6},$$

or $2 : 3 :: 4 : 6$ (read—2 *is to* 3 *as* 4 *is to* 6).

If we have several equal ratios we may denote the fact in this manner:—

$$1 : 2 : 3 : 5 = 4 : 8 : 12 : 20.$$

By the ratio of two lines we mean the ratio of their lengths, measured to any and the same scale.

We now bring before the student some further important fundamental properties of parallel lines.

Draw a straight line, and, to any scale, mark AB one unit, BC two units long. Through A, B, C draw any parallel straight lines. Let these parallels be cut by *any* other straight line abc. Measure ab, bc to any scale. Now verify by arithmetic (or by simply using the dividers) that

ab is one half of bc, just as AB was one half of BC,

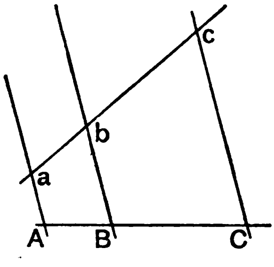
or $\text{ab} : \text{bc} = \text{AB} : \text{BC} = 1 : 2 = \frac{1}{2}$.

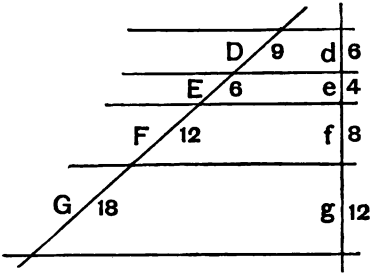
That is, however the transverse line abc be drawn, and whatever lengths the two segments ab, bc may be, the first is always half the second.

Instead of the parallels being as in the figure, let them be drawn at random, any distances apart. Let any transverse lines ABC, abc be drawn to cut them; measure, to any scale, AB, BC, ab, bc. Then by arithmetic verify

$$\text{ab} : \text{bc} = \text{AB} : \text{BC} = \text{constant}.$$

Further, let any number of parallels be drawn. Let





their successive distances apart be d, e, f, g, as shown. Let any line cut these parallels, being divided into segments of lengths D, E, F, G (to any scale). Verify that

$$D : E : F : G = d : e : f : g,$$

or

$$D : d = E : e = f : F = G : g = \text{constant}.$$

Examples.—**1.** In the figure above, the numbers are the lengths of d, e, . . . D, E, . . . in millimetres. Check these by using your centimetre scale. Then verify the following proportions by inserting the numbers and employing arithmetic :—

$$D : E = d : e\,;\quad E : F : G = e : f : g,$$
$$D : d = E : e = D + E : d + e = D - E : d - e,$$
$$F : G = f : g = F + f : G + g = F - f : G - g,$$
$$F : D + E + F + G = f : d + e + f + g.$$

2. Using any scale, measure the lengths of AB, BC, AC, ab, bc, ac, Aa, Bb, Cc in the figure above; then by arithmetic, as in Ex. 1, verify the following proportions :—

$$\mathrm{AB} : \mathrm{AC} = \mathrm{ab} : \mathrm{ac} = 1 : 3,$$
$$\mathrm{AC} : \mathrm{BC} = \mathrm{ac} : \mathrm{bc} = 3 : 2,$$
$$\mathrm{Bb} - \mathrm{Aa} : \mathrm{Cc} - \mathrm{Bb} = \mathrm{AB} : \mathrm{BC} = 1 : 2.$$

3. If $\frac{A}{a} = \frac{B}{b} = r$, verify by inserting numbers and using arithmetic that

$$r = \frac{A+B}{a+b} = \frac{A-B}{a-b} = \frac{mA+nB}{ma+nb}.\quad \text{Also that}$$

$$\frac{A+a}{A-a} = \frac{B+b}{B-b}\,;\quad \frac{A}{A+a} = \frac{B}{B+b}\,;\quad \frac{mA+na}{pA+qa} = \frac{mB+nb}{pB+qb},\ \text{where}$$

m, n, p, q are any numbers, whole or fractional, positive or negative.

37. PROBLEM.—(a) **To divide a line into parts in given proportions. (b) To measure the proportions of the parts of a given divided line.**

We shall indicate or suggest one or two ways in which the principles enumerated in the last article may be applied to the practical solution of the present problem. The student, however, will derive more pleasure and profit from any devices to effect the object which he may himself discover.

(a) *Let it be required to divide the given line* AB *into three parts*, AM, MN, NB, *which shall have the proportions of* 2 *to* 2.8 *to* 4.2.

From A draw a line making any suitable angle with AB, along which, to any convenient scale, mark off Am, mn, nb of lengths 2, 2.8 and 4.2 units. Join bB, and through n, m draw nN, mM, parallel to bB.

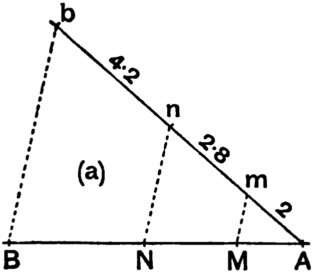
Or we might proceed as in Art. 32 (*e*), the scale being adjusted until a length of 2 + 2.8 + 4.2 or 9 units lay between the end perpendiculars. The points 2 and 4.8 would then be marked along the edge, and the solution completed by drawing the two intermediate perpendiculars through these points. Ruled or squared tracing-paper might sometimes be used.

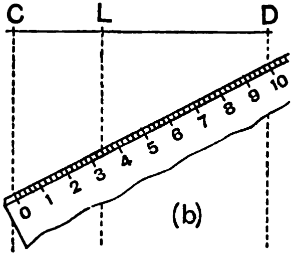
(b) *The line* CD *is divided into two parts at* L. *Find what fraction* CL *is of* CD. *Also what multiple* CD *is of* LD.

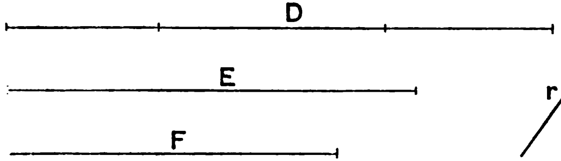
Through C, L, D draw any parallel lines as shown. Select a suitable scale, and adjust it so as to have the zero and unit (or the 0 and 10) points on the parallels through C and D. Then, between the parallels through C and L read off the required answer as a decimal fraction. In the figure this is 0.34.

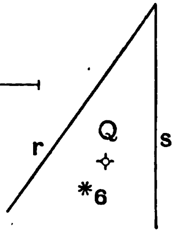
To find what multiple CD is of LD, a scale must now be adjusted with the 0 and 1 points lying on the parallels D and L. The result is then read off on the parallel C. The student should try this.

The data may be considerably varied. Thus, a line









may require to be divided similarly to a given divided line. After a little practice, examples of this nature are readily dealt with.

Examples.—**1.** Draw a line AB 2″ long. Divide it at C into two parts in the proportion AC : CB = 2.1 : 3.9. Also find a point D in BA produced, such that DA : DB = 2.1 : 3.9.

***2.** Divide a line 2.5″ long into parts in the same proportions as the parts into which the given line D above is divided. Obtain the result in three or four different ways. Let one of the methods be by the use of tracing paper.

***3.** Copy the lines D, E, F above. Then ascertain what fraction E is of D and what multiple E is of F. *Ans.* 0.75, 1.24. Let the student work this example in several ways.

***4.** Find a fourth proportional to the lines F, E, D, that is, find the length of a line X, such that F : E = D : X. *Ans.* 3.56″. Also find a fourth proportional to D, E, F. *Ans.* 1.28″.

5. Answer the following question by drawing and measurement:— If 3¼ lb. cost 5½d., what rate is this per lb.? *Ans.* 1.69 pence.

***6.** Draw a figure something like that shown above, but larger. Then through the point Q draw a line to meet the lines rs in R, S, such that RQ : QS = 2 : 3.

38. Systems of parallel lines.—The following propositions may be verified by actual measurement, but they will probably be regarded as self-evident.

(a) If two lines intersect one another, of the four angles thus formed, *opposite* angles are equal, and *adjacent* angles are supplementary. Thus in Fig. **(a)** the two angles marked α are equal, so are those marked β; and $\alpha + \beta = 180°$.

(b) If a line cut two parallel lines, *alternate* angles are equal, and adjacent angles are supplementary. Thus, in Fig. **(b)**, all the angles α are equal. So are the angles β; and $\alpha + \beta = 180°$.

(c) If two systems of parallel lines intersect, alternate angles are equal and adjacent angles are supplementary. Thus again, see Fig. **(c)**, all the angles marked α are equal. Also those marked β; and $\alpha + \beta = 180°$.

Note.—If one of the angles is 90°, then all the angles must be 90°.

39. Some properties of parallelograms.—Any figure with four straight sides is called a **quadrilateral**. A **parallelogram** is a quadrilateral in which **opposite sides** are **parallel**.

Thus in Fig. **(c)** the two systems of parallel lines give rise to a series of parallelograms. From the figure it is evident that **opposite angles** of a parallelogram are equal. And by measurement the student will easily satisfy himself that **opposite sides** are also **equal**.

The **diagonals** of a parallelogram are lines joining opposite corners, as A A, B B in the figure.

There are three important special forms of parallelograms, viz. :—

The **rhombus**, with its four sides all equal.

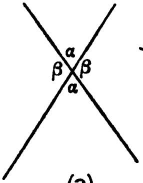
The **rectangle**, with the angles all equal (each 90°).

The **square**, with sides and angles all equal.

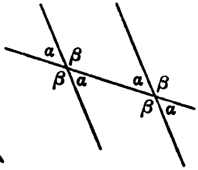
In order to become familiar with the parallelogram, let the student work the following examples by careful drawing and measurement.

Examples 1.—Construct accurately to scale :—

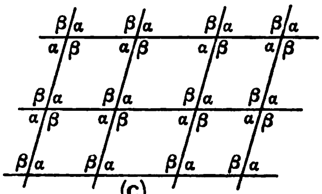
A square, 2″ side.



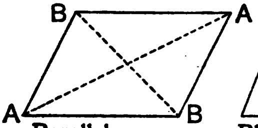
(a)



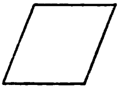
(b)



(c)

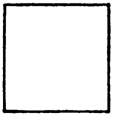


Parallelogram



Rhombus

Rectangle



Square

A rhombus, 2″ side, included angle 60°.
A rectangle, sides 2.5″ and 1.5″.
A parallelogram, sides 2.5″ and 1.5″, included angle 60°.

In each case ascertain by measurement—

(*a*) Whether the diagonals bisect one another.
(*b*) Whether the diagonals are at right angles.
(*c*) Whether a circle can be drawn to pass through all four corners.

2. (*a*) Can a quadrilateral be drawn having opposite angles equal and opposite sides unequal?
   (*b*) Can a quadrilateral be drawn having opposite sides equal and opposite angles unequal?
3. Cut four strips of paper or cardboard, and, using paper fasteners or other device, make a parallelogram which is hinged at the corners, some of the sides being prolonged. Mark any four points A, B, C, D in a straight line, one on each side or side produced. Verify that however the shape of this jointed parallelogram be altered, the four points always remain in a straight line, and that the ratios AB : BC : CD remain unaltered. See Ex. 2, Art. 57.

CHAPTER V

THE RIGHT-ANGLED TRIANGLE

40. Triangles in general.—The triangle, being one of the most important of geometrical figures, and its applications being numerous and far reaching, the student must carefully study its fundamental properties. We shall devote two chapters to this part of our subject, the present one dealing with right-angled triangles. We will first give, however, in this article, some preliminary definitions and explanations relating to triangles in general.

A **triangle** is a plane figure bounded by three straight lines, called its **sides**, meeting in three points, each of which is a **vertex** or **corner**. Three **angles** are formed by the sides taken in pairs.

Any side of a triangle may be regarded as its **base**, in which case the opposite corner is called **the vertex**, and the perpendicular from the vertex to the base is called the **altitude**. Corresponding with these terms we have the **vertical angle** and the two **base angles**.

There are varieties of triangles named as follows:—

An **equilateral triangle** has its sides all equal.

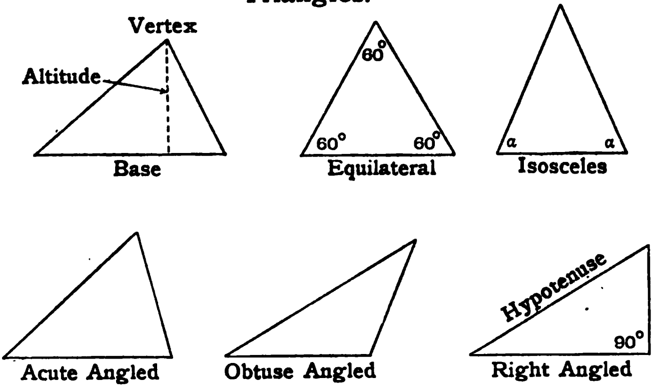
An **isosceles triangle** has two sides equal.

An **acute-angled triangle** has three acute angles.

An **obtuse-angled triangle** has one obtuse angle.

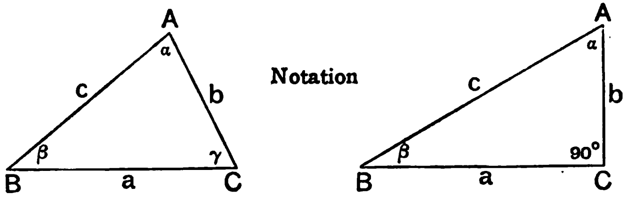
A **right-angled triangle** has one right angle. The side opposite the right angle is called the **hypotenuse**. If the two perpendicular sides are horizontal and vertical they

Triangles.



Base Equilateral Isosceles

Acute Angled Obtuse Angled Right Angled



Notation

may be called the **base** and **height** respectively. And the acute angles at the base and vertex may be called the **base angle** and the **vertical angle** respectively.

Notation.—Let A, B, C be the vertices of any triangle. The same italic letters A, B, C may also be used to denote the magnitudes of the three angles, and the corresponding small letters a, b, c may stand for the lengths of the opposite sides. See the figure. Sometimes it will be convenient to use the Greek letters α, β, γ (alpha, beta, gamma) for the angles. The three angles and the three sides are called the **elements** of the triangle. Thus in any triangle there are **six** elements.

41. Some properties of right-angled triangles.—We shall here give two important properties which characterise right-angled triangles. First:—

(a) The two acute angles are complementary.—Their sum is always 90°. The student knows this to be true in the case of his set-squares, which afford good illustrations of right-angled triangles. He may verify it by measurement in other cases. In the figure the angles α and β measure 62° and 28°. So $\alpha + \beta = 62^\circ + 28^\circ = 90^\circ$. And $\alpha = 90^\circ - \beta$; $\beta = 90^\circ - \alpha$.

He may give some reason why this proposition is true. For example let BP be perpendicular to BC; then BP is parallel to CA, and so the angles marked α are equal. See Art. 38. Thus $\alpha + \beta = 90^\circ$ always.

The student should notice that $\sin \alpha = \cos \beta$, and $\cos \alpha = \sin \beta$.

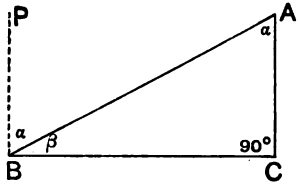
Verification by paper folding.—The following is instructive. Cut out any right-angled triangle ABC in paper. Fold the paper so as to make A and B coincide with C. It will be found that the figure folds into the form of the rectangle OMCN, and that α and β together just make up the angle C, which is 90°. It further appears that OA, OB and OC are all equal, so that if a semicircle (centre O) were described on AB, it would pass through C. This fact should be remembered.

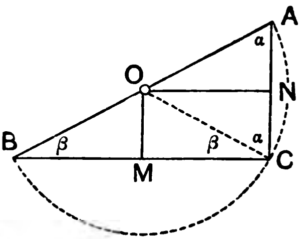
(b) The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the two sides.—This is the famous theorem of Pythagoras, and it is one of the most important in elementary geometry. Letting a, b, c stand for the lengths of the sides and hypotenuse, this proposition may be thus expressed:—

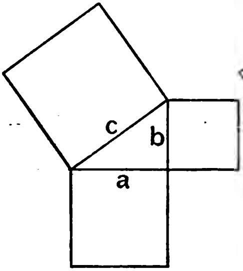
$$a^2 + b^2 = c^2;\ \text{or}\ a^2 = c^2 - b^2;\ \text{or}\ b^2 = c^2 - a^2.$$

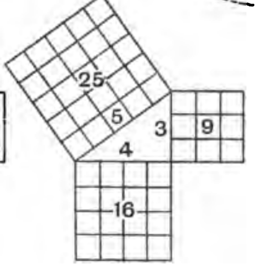
The two latter forms interpreted would read:—*The square on one side is equal to the difference between the squares on the hypotenuse and the other side.*

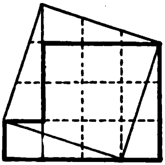
We have already, in Art. 29 (*e*) used a right-angled triangle whose three sides were 3, 4, and 5 units long. In











this case the student at once verifies that $3^2 + 4^2 = 5^2$, or $9 + 16 = 25$. See also the figure above. Let him draw several different right-angled triangles, also a rectangle with the two diagonals, and by careful measurement and arithmetic, verify this important property.

Verification by paper cutting.—The last figure shows an interesting and well-known method by which the large square may be cut into three parts and rearranged so as to form the two smaller squares, or *vice versa.* We leave this as an exercise for the student.

Examples.—**1.** Describe any circle, and draw any diameter AB. Take any point C on the circumference, and join CA, CB. Verify that ABC is a right-angled triangle.

2. Draw any square, say of 2″ side. Then draw two other squares of double and half the area of the first square.

3. Verify that in any right-angled triangle $\sin^2 \alpha + \sin^2 \beta = 1$. Also that for any angle θ, $\sin^2 \theta + \cos^2 \theta = 1$.

4. If CM be a perpendicular on the hypotenuse of a right-angled triangle, verify that $AM : BM = AC^2 : BC^2$. Also that $CM^2 = AM . BM$.

5. Draw a square of 1.5″ side; then draw squares of three and four times the area of the first one.

42. Construction of right-angled triangles. The nine cases—

We now show how to draw or solve a right-angled triangle having given any two independent elements. There are *five* elements, exclusive of the right angle, *viz.*—The base a, the altitude or height b, the hypotenuse c, the base angle B, and the vertical angle A.

Distinguishing the perpendicular sides as base and height, there are nine possible variations of the data. The linear scale for the figures is $\frac{1}{4}$.

I. Given the base and height. Say 3.5″ and 2″.

Draw the base BC of length 3.5″, and the perpendicular CA of length 2″. Join AB.

II. Given the base and hypotenuse. Say 3.5″ and 4″.

Draw the base BC of length 3.5″ and draw a perpendicular from C. With centre B, radius 4″ cut this perpendicular in A. Join AB.

III. Given the height and hypotenuse. Say 2.5″ and 4.5″.

Draw the base line, and the perpendicular CA 2.5″ long. With centre A, radius 4.5″ cut the base in B. Join AB.

IV. Given the base and base angle. Say 3.6″ and 36°.

Draw the base BC 3.6″ long, and the perpendicular from C. Draw BA making 36° with BC.

V. Given the height and base angle. Say 1.8″ and 29°.

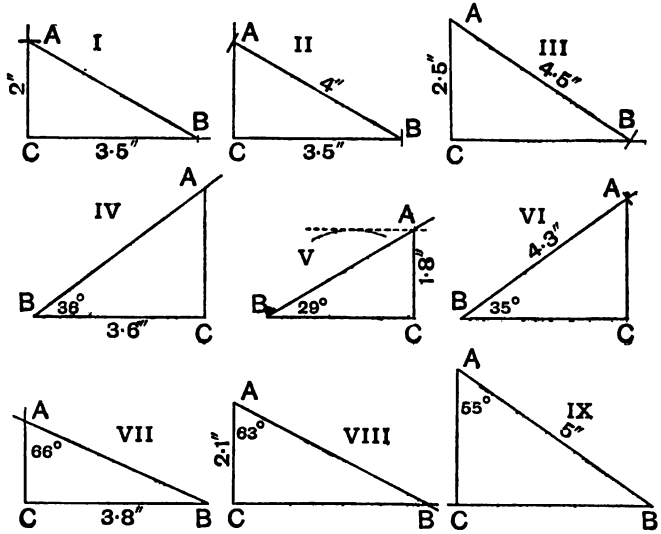
Draw the base line, and a line parallel to it at distance 1.8″. Draw the hypotenuse making 29° with the base, and cutting the parallel in A. Draw the perpendicular AC.

VI. Given the hypotenuse and base angle. Say 4.3″ and 35°.

Draw the base line. Draw the hypotenuse BA making 35° with the base, and 4.3″ long. Draw the perpendicular AC.

VII. Given the base and vertical angle. Say 3.8″ and 66°.

Draw the base BC of length 3.8″, and draw a perpendicular at C. Draw the hypotenuse BA making 90° − 66° or 24° with BC.



VIII. Given the height and vertical angle. Say 2.1″ and 63°.

Draw the base line, and the perpendicular CA of length 2.1″. Draw the hypotenuse AB making 63° with AC.

IX. Given the hypotenuse and vertical angle. Say 5″ and 55°.

Draw a vertical line, and the line AB making 55° with it and 5″ long. Draw the base BC.

Examples.—Solve the following nine right-angled triangles by accurately drawn figures, and measure the results :—

1. Given $a=3''$, $b=2''$. *Ans.* $c=3.16''$, $B=33.7^\circ$.
2. Given $a=1.7''$, $c=4''$. *Ans.* $b=3.62''$, $B=64.9^\circ$.
3. Given $b=2.1''$, $c=3''$. *Ans.* $a=2.14''$, $B=44.4^\circ$.
4. Given $a=2.9''$, $B=35^\circ$. *Ans.* $b=2.03''$, $c=3.54''$.
5. Given $b=3.2''$, $B=66^\circ$. *Ans.* $a=1.42''$, $c=3.50''$.
6. Given $c=2.4''$, $B=9^\circ.1$. *Ans.* $a=2.37''$, $b=0.38''$.
7. Given $a=2.3''$, $A=75^\circ$. *Ans.* $b=0.62''$, $c=2.38''$.
8. Given $b=1.3''$, $A=28''$. *Ans.* $a=0.69''$, $c=1.47''$.
9. Given $c=3.4''$, $A=33''$. *Ans.* $a=1.86''$, $b=2.85''$.

43. General examples and applications of right-angled triangles.—As we proceed with our subject it will be seen that many of the problems which occur resolve themselves into problems on right-angled triangles. The nine standard cases of the last article by no means exhaust all the varieties in the conditions. There may be other kinds of data besides the simple elements. Moreover, the triangle may be required to occupy some definite position as well as have a definite shape. For present illustration let the student work the following examples:—

Notation.—*In a right-angled triangle* ABC, C *will be the right angle, and* CM *the perpendicular on the hypotenuse* AB.

1. Find the length of the longest line that can be drawn on an imperial sheet of drawing-paper, 30" × 22", or on the floor of a room 30' × 22'. Obtain the answer both graphically and by calculation.
2. Draw any rectangle of large size. Then test its accuracy by measuring the distances apart of its corners. How many measurements would you make?
3. Construct a right-angled triangle having given—hypotenuse AB = 2.6"; perpendicular, CM = 1.1". *Hint.* Draw a semi-circle on AB.
4. Draw a right-angled triangle ABC, having given—$\sin A = 0.37$, BC = 1.25".
5. In any right-angled triangle ABC, verify that
   Base = hypotenuse × cosine of base angle, *i.e.* $a = c \cos B$.
   Height = hypotenuse × sine of base angle, *i.e.* $b = c \sin B$.
   Height = base × tangent of base angle, *i.e.* $b = a \tan B$.
6. A vertical flag-pole rising from level ground casts a shadow 65 feet long. The shadow cast by a vertical rod 6 feet high is 7.3 feet long. Determine the angle of elevation of the sun above the horizon and the height of flag-pole. *Ans.* 39.5°, 53.5 feet.
7. Two brick walls, CE, CN, 7 feet high, meet at right angles, the directions of CE and CN being East and North. At a certain hour in the afternoon the shadows of the two walls extend 2.3 feet from the base of the wall CE and 10.4 feet from the base of the wall CN. Find the direction of the sun in degrees West of South, and the angle of elevation of the sun above the horizon. *Ans.* 11.9°, 33.4°.
8. The distance between two places A and B is 27 miles. The direction from A to B is E.37°N. Find how much B is

to the East and how much to the North of A. *Ans.* 27 cos 37° and 27 sin 37°, or 18.1 and 16.2 miles.

9. In order to determine the height of an electric arc light above a level floor, a rod PQ 80″ long was held vertically with its lower end P resting on the floor. Its shadow Pq was measured and found to be 61.5″ long. The point q on the floor was marked and the rod transferred to q, being again held vertically. Its shadow now measured 88″. Find the height of the light; its horizontal distance from q; and its actual distance from q. *Ans.* 23′-2″, 18′-4″, 29′-7″.

10. You are given that in a right-angled triangle ABC (see Figs. p. 49), the base BC is 3″ long and the height CA is 2″. Solve the triangle by *calculation.* That is, determine the hypotenuse and the angles *A* and *B*.

Solution.—The theorem of Pythagoras, Art. 41 (*b*), tells us that

$$AB^2 = AC^2 + BC^2 = 2^2 + 3^2 = 4 + 9 = 13.$$

$$\therefore\ AB = \sqrt{13} = 3.61''.$$

Again, we have (see p. 22),

$$\tan B = \frac{AC}{CB} = \frac{2}{3} = 0.667.$$

And referring to the Table of Tangents on p. 24, we find that the angle whose tangent is 0.667 is an angle of 33.7°.

$$\therefore\ B = 33.7^\circ.$$

Lastly, since *A* is the complement of *B*, see 41 (*a*), we have

$$A = 90^\circ - B = 90^\circ - 33.7^\circ = 56.3^\circ,$$

or thus

$$\tan A = \frac{BC}{CA} = \frac{3}{2} = 1.5.$$

That is, $A = \tan^{-1} 1.5 = 56.3^\circ$, by the tables, p. 24.

Note.—The symbol "$\tan^{-1} 1.5$" is shorthand for "the angle whose tangent is 1.5."

11. Solve the nine examples on p. 49 by calculation, after the manner of Ex. 10, and so verify the measured results.

Note.—As suggested in Examples 10 and 11, the student at this stage of his progress should begin the *numerical* solution of right-angled triangles; and throughout his subsequent work, graphical and numerical solutions should go on together, mutually illustrating each other. It is surprising what a large proportion of calculations involving angles reduce to merely solving right-angled triangles. The opportunity afforded of becoming *perfectly familiar* with such triangles under all sorts of conditions, and also with trigonometrical tables, is far too valuable a one to be missed.

CHAPTER VI

THE TRIANGLE. SIMILAR TRIANGLES

44. Some general properties of triangles.—Let the student refer to Art. 40 for the explanation of a number of terms and a system of notation used in connection with triangles. We here give some fundamental properties of triangles.

(a) Any two sides of a triangle are together greater than the third side.—Or, *Any side of a triangle is greater than the difference between the other two.* In symbols

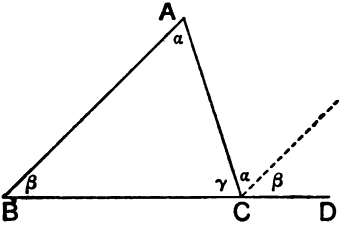
$$(a + b) > c, \quad \text{or } a > (c - b).$$

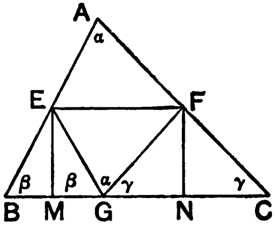
In illustration let the student work Examples 1, 6, and 7, p. 55. The proposition stated in the first form may be regarded as self-evident since we know from common experience that the straight or direct path connecting any two points is shorter than any other.

(b) The three angles of any triangle are together equal to two right angles, or $A + B + C = 180°$.—In verification let any triangle (of fairly large size) be drawn, and its three angles measured with a protractor, and then added. This is worth doing very carefully; the sum should be exactly 180°.

Again, let the three angles be added graphically, using tracing-paper. The proposition should again be confirmed.

Proof by deduction.—Let ABC be a triangle. Produce BC to D and draw CP parallel to BA. Then by the properties of parallel lines, the two angles marked a are





equal; so are those marked β. See Art. 38. Thus $\alpha + \beta + \gamma = 180^\circ$, that is $A + B + C = 180^\circ$.

Verification by paper folding.—This is interesting and instructive. Cut out the triangle ABC in paper. Bisect (by folding) AB, AC in E, F. Fold along EF, bringing A to the position G. Then by two more foldings bring B and C to G. The triangle is thus folded into a rectangle EFNM, the angles α, β, γ being brought together at G, and these should just make up 180°.

(c) **The exterior angle of any triangle is equal to the sum of the two interior and opposite angles.**—The *exterior angle* at a vertex of any triangle (or polygon) is the angle between one of the sides and the other side produced. Thus ACD is an exterior angle at C. The truth of the above proposition is evident from the figure, since ACD $= \alpha + \beta = A + B$. Moreover, the exterior angle is the supplement of the adjacent interior angle, or ACD + ACB $= 180^\circ$.

(d) **Congruent triangles.**—The propositions of Euclid I. concerning identically equal or congruent triangles, are equivalent to affirming that the shape and size of a triangle is defined when any **independent** three of its elements are specified, such as the three sides, or two sides and the included angle, etc.; but not the three angles, since their values cannot be assigned all independently, their sum being 180°. These propositions are illustrated in the next article, which deals with the plotting of triangles from given data. Let the student draw these triangles to the given dimensions, scale $\frac{1}{2}$.

45. Construction of triangles. The five standard cases.

I. Given the three sides a, b, c. **Say 4.6″, 3.4″, 2.7″.**

Draw one side, say BC, of the given length *a*, or 4.6″.

With centre B, radius *c* or 2.7″, describe an arc. With centre C, radius *b* or 3.4″, cut this arc in A. Join AB, AC.

Limitations.—The sum of any two sides must be less than the third side. *Note.*—The scale for the figures is $\frac{1}{4}$.

II. Given two sides and the included angle, say a, b, C; **3.6″, 3″, 81°.**

Draw the side BC of the given length *a*, or 3.6″. From C draw CA at *C*° or 81° to CB, and of length *b* or 3″. Join AB.

Note.—There are no limitations to the data.

III. Given two angles and the common side. Say B, C, a; **52°, 93°, 3.2″.**

Draw the side BC of the given length *a* or 3.2″.

From B draw the side BA, making *B*° or 52° with BC. From C draw the side CA, making *C*° or 93° with CB.

These two sides will meet in A.

Limitation.—$B+C$ must be less than 180°.

IV. Given two sides and an opposite angle. Say a, b, B; **4.6″, 3.2″, 35°.**

Draw the side BC of the given length *a* or 4.6″.

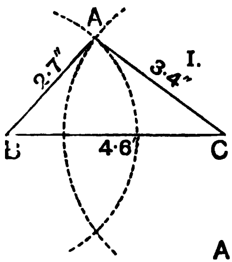
From B draw a line making *B*° or 35° with BC. With centre C, radius *b* or 3.2″, describe an arc cutting this line in A, A′. Join AC, A′C.

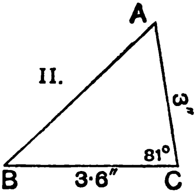
Then the triangles ABC, A′BC both comply with the data.

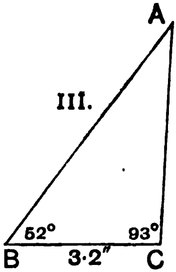
Note.—There are thus seen to be two solutions, and this is known as the **ambiguous case.**

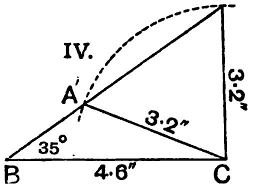
Limitations.—If CA is equal to the perpendicular from C on BA, *i.e.* if $b=a \sin B$, there is only one solution—a right-angled triangle. If *b* is less than $a \sin B$ the triangle is impossible.

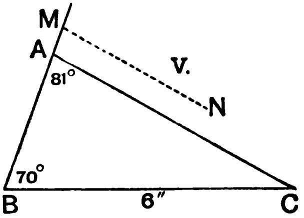
V. Given two angles and an opposite side. Say A, B, a; **81°, 70°, 6″.**











Draw the side **BC** of the given length a or 6″. Through **B** draw a line making $B°$ or 70° with **BC**. Take any point **M** on this line and draw **MN** at $A°$ or 81° to **MB**. Draw **CA** parallel to **NM**.

Limitation.—$A+B$ must be less than 180°.

Examples.—Solve the following five triangles having given :—

1. $a=4''$, $b=3.5''$, $c=3''$. *Ans.* $A=75.5°$, $B=57.9°$.
2. $a=2.7''$, $b=3.8''$, $C=113°$. *Ans.* $c=5.46''$, $A=27.1°$.
3. $a=2.3''$, $B=7°$, $C=157°$. *Ans.* $b=1.02''$, $c=3.26''$.
4. $a=3.6''$, $b=2.8''$, $B=51°$. *Ans.* $c=2.27''$, $A=90°$.
5. $a=3.4$ $A=75°$, $B=62°$. *Ans.* $b=3.11''$, $c=2.41''$.

Construct if possible the following eight triangles to the given data. If any triangle is impossible, give the reason :—

6. $a=4''$, $b=2''$, $c=1.5''$.
7. $a=4''$, $b=5''$, $c=0.5$.
8. $a=1''$, $b=3''$, $C=2°$.
9. $a=1''$, $b=3''$, $C=178°$.
10. $a=3''$, $B=80°$, $C=120°$.
11. $a=4''$, $b=2''$, $C=30°$.
12. $a=4$, $b=1.5,''C=30°$.
13. $a=2$, $A=75°$, $B=135°$.

46. Further properties of triangles.—Let the student, in any way he pleases, convince himself of the truth of the following propositions:—

(a) *In any triangle the vertical angle is equal to either of the external angles at the base, minus the other (internal) base angle.*

(b) *In any triangle a greater side is subtended by a greater angle, and a greater angle by a greater side.*

(c) *If the three sides of a triangle are equal, so are the three angles. And conversely, if the angles are equal, so also are the sides.*

(d) *If two sides of a triangle are equal, the angles opposite these sides are also equal. And conversely.*

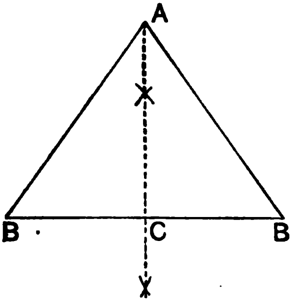
(e) *In any isosceles triangle the perpendicular bisector of the base contains the vertex, and is an axis of symmetry.*

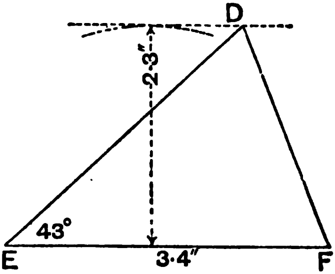
We have drawn a figure for this case. A is the vertex, B B the base, and A C the perpendicular bisector or **axis of symmetry**. Let the student copy this triangle on tracing-paper, and fold the paper along A C; he will find that the parts on each side the axis will come together and entirely coincide, forming one figure in duplicate. This illustrates the meaning of the term axis of symmetry or **axis**. If any other point A′ were taken on the axis, then A′B B would be an isosceles triangle.

In addition to the six elements of a triangle, there are other magnitudes connected with it which would serve as alternative data in plotting the triangle, such for example as the altitude, the **perimeter** or sum of the sides, the sum or difference of two sides or two angles, etc. We will now give an example in illustration, but the student should rely on his general knowledge and ingenuity in dealing with any special cases that may arise.

47. PROBLEM.—**To construct a triangle having given the base 3.4″, one base angle 43°, and altitude 2.3″.**

Draw the base E F 3.4″ long. From one end E draw a line making 43° with E F; then the vertex must lie on this line. And draw a parallel to E F at a distance of 2.3″; the vertex must also be in this parallel. These two lines intersect in the required vertex D. Join D F.





Examples.—**1.** Draw an equilateral triangle of 2.7" side. Also one having an altitude of 2.7".

2. Construct the following isosceles triangles. having given :—

(*a*) Base 1.5", each side 2".
(*b*) Base 1.9", altitude 1.1".
(*c*) Base 3", vertical angle, 150°.
(*d*) Vertical angle, 50°, each side 2".
(*e*) Altitude 2", vertical angle half base angle.
(*f*) Altitude 2", perimeter 5".

3. Construct the following triangles, having given :—

(*a*) Base 2.5", one side 2", altitude 2".
(*b*) Base angles 50° and 70°, altitude, 2".
(*c*) Altitude 2", sides 3" and 2.5".
(*d*) Side $a = 2''$, ratio of angles $B : A : C = 3 : 4 : 5$.

4. Draw any triangle ABC. Then copy the triangle in a new position :—

(*a*) When a given side has any given situation.
(*b*) When it is turned about one corner through a given angle, say 30°.
(*c*) When it is turned about one corner through 180°.
(*d*) When it is turned over about one side.
(*e*) When it is turned about any point in its plane through any angle.

N.B.—In this example tracing-paper and pricking through may be used with advantage as an alternative to compasses.

5. Draw any triangle. Draw lines from the three corners perpendicular to the opposite sides. Are these three lines always concurrent? There are other systems of three concurrent lines; try and discover some of them.

48. Similar triangles.—In Art. 45 it was shown how to plot any triangle, having given three elements. There were five cases considered, but the student may have observed that one case was missing, *viz.* that in which the three given elements were the three angles.

Let him now examine this case, and draw a triangle such that its angles shall be say 75°, 60°, 45° (sum 180°). He will find that he can draw an indefinite number of such triangles, larger or smaller. The triangles, however, will have this common property, that any one is a magnified or reduced image or copy of any other. Or otherwise stated, all the triangles may be regarded as drawings representing the same triangle but to different scales. Let him measure and compare the sides of any pair of these triangles. He will find that corresponding sides are proportionals. Two of these triangles are shown, the scales being such that corresponding sides are in the ratio of 2 to 3.

Such triangles are said to be **similar**, the characteristic feature of similar figures being that *corresponding angles are equal, and corresponding lines are proportional.* One figure is a reduced or enlarged copy of the other.

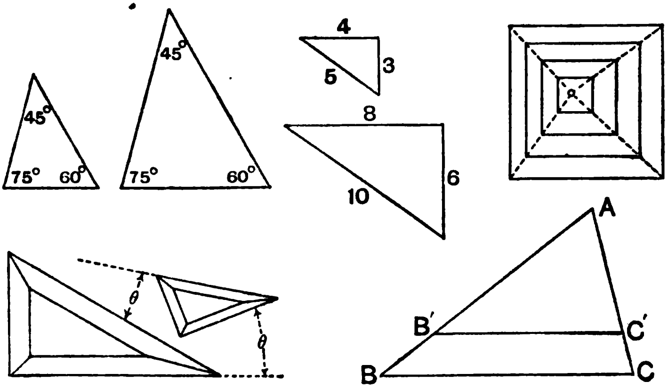
Thus all equilateral triangles are similar figures. So are squares, or circles, or cubes, or spheres. Right-angled triangles having the sides in the ratios 3 : 4 : 5 are all similar; so are isosceles triangles with equal vertical angles.

Draw any triangle ABC, and draw B′C′ parallel to BC as in the figure. Then ABC, AB′C′ are similar triangles, being equiangular, and having

$$\mathrm{AB:BC:CA=AB':B'C':C'A},$$

or $\mathrm{AB:AB'=BC:B'C'=CA:C'A}=4:3$ in the figure.

In similar figures, the lines of one are not necessarily parallel to the corresponding lines of the other. They may be inclined to one another at any constant angle θ, just as if one figure had been turned through this angle relatively to the other. See the illustration of the two set squares.



Similar Figures.

Examples.—In some of these examples the following general method of attacking problems will be of service :—

If a required figure cannot be at once plotted, endeavour to draw one which shall be similar. If successful, enlarge or reduce this figure to scale, and thus obtain the answer.

1. Draw any triangle ABC. Then reduce it in size so that the sides are half their original lengths. Also enlarge it so that the sides are now $1\frac{1}{2}$ times their original lengths. And draw a triangle similar to ABC, corresponding sides of the two triangles being proportional to two given lines. See Art. 37.

2. Draw any triangle ABC and mark any two points A′, B′. Draw an enlarged or reduced copy of the triangle, such that the side AB shall be situated at A′B′.

3. Draw a triangle having the sides a, b, c in the ratio 4 : 3 : 2. (**a**) When $a = 3.5''$; (**b**) when the altitude (base a) is $2.5''$; (**c**) when the perimeter is 7 inches.

4. Draw any triangle ABC. Measure its sides a, b, c and also its angles A, B, C. Take out the sines of these angles from the tables, p. 24. Then by arithmetic verify that the sides are proportional to the sines of the opposite angles, viz.—

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

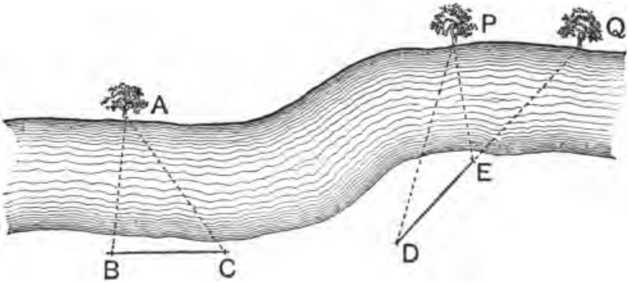
Also verify that $a^2 = b^2 + c^2 - 2\,bc \cos A$, or $\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}$.

49. Applications.—The applications of the triangle in the Arts are many and important, and we shall conclude this chapter with some general remarks on the subject, and some miscellaneous examples.

One important function of the triangle is its use in locating points. Some surveying operations are largely based on this property. The operator uses chains and rods for measuring lengths; and instruments, such as the sextant and theodolite, for measuring angles. Starting from a given measured base, a distant object is located by measuring the angles which its directions at the two ends of the base make with the base. This point then becomes part of a new base from which a more distant point is located. Proceeding in this way, a tract of country to be mapped is divided into a network of triangles; other methods are used for filling in details. In the great Ordnance Survey of this country made half a century ago, one of these base lines was on Salisbury Plain, and was nearly 7 miles long. Another was near Lough Foyle in Ireland, and was nearly 8 miles long. The former was calculated by triangulation, starting from the Lough Foyle base, and the result only differed 5 inches from the actual measured length. In a survey of this magnitude allowance had to be made for the curvature of the earth's surface.

In measuring the distance of the sun or planets, the ends of the base are at opposite sides of the earth, some 8000 miles apart. In estimating the distance of a fixed star, the base is about 190 millions of miles long, being a diameter of the earth's orbit round the sun. The nearest of the fixed stars is at such an enormous distance away that even with this immense base the lines from the star to the ends of the base include an angle of only about $\frac{1}{2000}$ of a degree, corresponding to a distance of over 20,000,000,000,000 miles.

Coming to more homely matters, the figure shows how the width of a river could be ascertained by a person on one bank. He might select and measure a base BC. Then note some conspicuous object A on the remote bank, and by a sextant or other instrument measure the two angles



ABC, ACB. The triangle ABC could then be plotted, and the width measured.

An alternative method of procedure would be:—From some convenient place D measure the angle PDQ subtended by two visible objects on the opposite bank. Then walk a measured distance DE in the direction of one of the objects, say Q, and from E measure the angle PEQ now subtended by P and Q. The triangle PDE could be drawn from this data, and the distance EP measured.

If the student have time, he would find it instructive to design and fit up some primitive apparatus for measuring angles, and armed with this and a tape measure do a little amateur surveying on his own account. Let the reader refer to Art. 204.

Another property of the triangle frequently taken advantage of is that its shape cannot alter if its sides remain of the same length. This is applied by engineers and architects in constructing braced frames built up of triangles, such as cranes, roof trusses, and bridges for carrying loads. The triangle is also used in finding the forces to which the bars of these frames are subject, as will be seen in a future chapter.

As a last example, a large number of decorative designs have a geometrical basis, and in the building up of such figures the triangle plays an important part. See p. 13.

50. Miscellaneous Examples.

1. Draw any triangle ABC; bisect the sides BC, CA, AB in D, E, F. Join AD, BE, CF. Observe that these three lines all intersect one another in the same point G.

   Note.—The point G is the **centre of gravity** of the triangle. Cut out a model of the triangle in cardboard and suspend it by a thread from the point G. Also suspend the triangle in succession from any two other points, and in each case mark the line of the string. These lines produced should intersect in G.

2. Draw a triangle ABC, sides say 3″, 2.7″, 2.4″. Then draw (*a*) the perpendiculars from the corners on to the opposite sides, (*b*) the bisectors of the angles, (*c*) the perpendicular bisectors of the sides. Note that in each case the three lines are concurrent.

3. Draw any triangle ABC and determine (*a*) the circle touching all the sides (called the **inscribed circle**), and (*b*) the circle passing through the three angular points ABC (called the **circumscribing circle**).

4. Draw two lines nearly parallel, and which would intersect it produced at a point P out of reach. Mark a point A on one of the lines, and by construction, measurement, and arithmetic determine the distance of P from A. Also draw the bisector of the angle between the lines.

   Hint.—Make use of the properties of similar triangles.

5. Suppose it were required to find the distance apart of two points on the floor of a room between which was an obstacle preventing direct measurement. Show several ways in which you could do this if you were provided with a tape-measure and a chalked string, but not a set-square. Show also a method in which the latter would be required.

6. In the figure on p. 61 find BA if BC = 67 feet, ABC = 86°, ACB = 75°. *Ans.* 199 feet. Also find EP if DE = 43 yards, PDQ = 28°, PEQ = 51° *Ans.* 51.6 yards.

7. Draw any triangle, and determine the three **escribed circles**, that is, the circles outside the triangle, each of which touches one side and the other two sides produced.

8. Two inaccessible objects P, Q as seen from a place A subtend an angle PAQ of 90° From a second place B the angle PBQ subtended is also 90°. From a place C, midway between A and B, the subtended angle PCQ is 97°. If the distance between A and B is 56.3 yards, what is the distance between P and Q? *Ans.* 118 yards.

9. Draw a rectangle ABCD, sides AB = CD = 3″, BC = AD = 4″. Find a point O inside the rectangle such that the angles AOB,

BOC subtended by AB and CD at O are both 90°. Measure the distance of O from AB and BC. *Ans.* 1.44″, 1.92″.

Also find a point P in the side AD such that the angles APB, BPC subtended by AB and BC at P are equal to one another. Measure this angle. *Ans.* 65.2°.

10. Construct a triangle ABC having given base $a=7.6$ cm.; perimeter $a+b+c=11.2$ cm.; base angle $B=36°$.

11. In a triangle ABC, AD is the perpendicular on BC; AB is 3.25 feet; the angle B is 55°. Find the length of AD. If BC is 4.67 feet, what is the area of the triangle?

Find also BD and DC and AC. Your answers must be right to three significant figures.

This example is taken from the Examination Paper in Practical Mathematics for 1902. It suggests how in Case II., p. 54, the triangle can be solved numerically by a student who can deal with right-angled triangles. In regard to the area see Art. 60. This kind of calculation can with great profit accompany graphical construction. Any triangle is readily solved numerically by being split up into two right-angled triangles, without requiring special rules. Let the student work the next example, No. 12.

12. After having drawn the triangles of Exs. 1 to 5, p. 55, to scale, and measured the results, verify the answers by numerical calculation, using the tables on p. 24.

We suggest the method for Ex. 1. See Fig. 1, p. 55.

Draw AD perpendicular to BC. Let x be the length of BD, and therefore $a-x$ the length of DC.

In the right-angled triangles ADB, ADC, we have

$$AD^2=AB^2-DB^2=AC^2-DC^2.$$

That is $$c^2-x^2=b^2-(a-x)^2=b^2-(a^2-2ax+x^2).$$

From this equation x or BD can be calculated. Then the two right-angled triangles are readily solved.

CHAPTER VII

CONSTRUCTION OF POLYGONS. COPYING TO SCALE

51. Some general explanations and propositions.—The word **polygon** (many-sided) is used to denote any figure which is bounded by straight lines—its sides. The number of sides may be three or upwards.

A **triangle** has **three** sides.
A **quadrilateral** has **four** sides.
A **pentagon** has **five** sides.
A **hexagon** has **six** sides.
A **heptagon** has **seven** sides.
An **octagon** has **eight** sides.

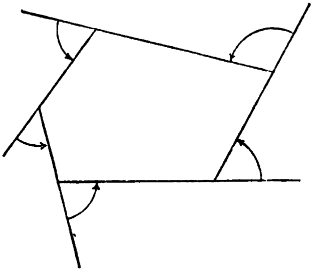
In what follows, unless otherwise stated, by polygon is to be understood a **convex** polygon, all the internal angles being less than 180°.

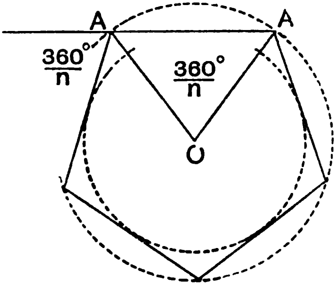
A **regular polygon** is one which is **equilateral** (equal sides), and also **equiangular** (equal angles), such for example as a square. It has a **centre** O, equidistant both from the corners and sides, so that circles can be **inscribed** in, and **circumscribed** about a regular polygon.

The following property is common to all polygons:—

Theorem 1.—*In any polygon the external angles are together equal to four right angles. Also the angles subtended by the sides at any internal point make up* 360°.

The second part of this theorem is obvious. And the first part becomes evident if we suppose a journey to be made round the polygon:—On arriving at the corners the traveller turns through the external angles in succession, during which he "boxes the compass,"





making one complete turn of 360°. See the figure. The next theorem is a particular case of Theorem I.

Theorem 2.—*In any regular polygon of n sides, the exterior angles, and the angles subtended by the sides at the centre are all equal, and each is* $360° \div n$.

Construction of regular polygons.—In Art. 13 we have shown a method of proceeding which on account of its accuracy and convenience should be adopted whenever possible. To meet other cases we give the following two constructions, but students would do better to invent methods of their own :—

(a) PROBLEM.—**On a given side AA to construct a regular polygon of n sides.**

On AA as base draw an isosceles triangle having a vertical angle of $360° \div n$ (base angle $= 90° - 180° \div n$). With centre O draw the circumscribing circle. Then step off AA n times round the circle and draw the sides.

(b) PROBLEM.—**In a given circle to inscribe a regular polygon of n sides.**

Do this by trial with the dividers. *Or* draw two radii OA, OA containing $360° \div n$. Then step off AA n times round the circle and draw the sides.

Example. Calculate the sum of the internal angles in any polygon of n sides. *Ans.* $(n-2)$ 180°. Also in any quadrilateral. *Ans.* 360°. See also Exs. 4 and 5, p. 69.

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52. The quadrilateral.—We have in Art. 39 considered the parallelogram, including as particular cases the rhombus, rectangle and square. We now take up the general case of the quadrilateral, and consider some other of its special forms.

First let the student draw any irregular quadrilateral ABCD. Now let him take measurements of its sides, diagonals, or angles, just sufficient to define its shape. He may sketch the figure (not to scale), insert the dimensions, and then see if he can reproduce the original figure from the information conveyed by the dimensioned sketch.

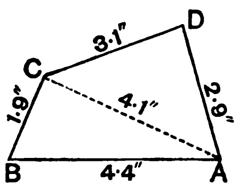
Question.—How many dimensions are necessary and sufficient in this case? *Answer.* Five. These might be the four sides and one diagonal, as in the figure, or four sides and an angle, or three sides and two diagonals, or other alternative combinations.

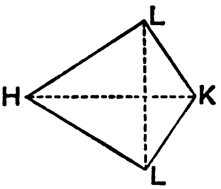
Triangulation.—Suppose the quadrilateral to be divided into two triangles by drawing one of the diagonals AC. Three dimensions would define the shape of one of the triangles ABC. Two more would locate the point D; making five dimensions in all. If the figure had been a pentagon, two additional dimensions would define the position of the additional point; total, seven. And generally, for any irregular polygon of n sides, we must know $2n - 3$ independent quantities, lengths or angles, in order to be able to plot its shape to scale.

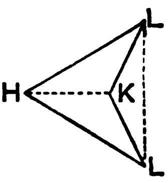
The kite.—This is a quadrilateral having pairs of adjacent sides equal. Two forms of the kite are shown in the figure, one having a re-entrant angle at K. In both cases HL = HL and KL = KL.

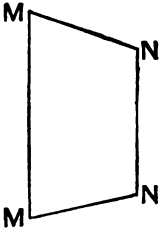
Let the student by measurement, paper folding, superposition, deduction, or in any manner he pleases, establish the fundamental property of the kite, viz.: *The diagonal* HK *is an axis of symmetry, and bisects the diagonal* LL *at right angles.* A knowledge of the properties of the kite is useful in deductive geometry.

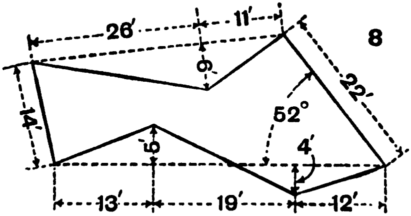
The trapezoid.—This is a quadrilateral having two opposite sides parallel. See the sides MM, NN in the figure. Its applications will be seen later.











Examples.—**1.** Draw a rectangle, diagonal 3″, one side 2″.

2. Draw a rhombus, diagonals 3″ and 1.8″ long.

3. Draw a parallelogram, sides 3″ and 2″ long, one angle between diagonals 45°.

4. Draw a kite having given, LL = 2″, LH = 2.2″, LK = 1.3″. Measure HK and the angles H, L, K. *Ans.* HK = 2.79″ or 1.13″, $H=54^\circ$, $L=103^\circ$ or 23°, $K=100^\circ$ or 260°.

5. Draw a trapezoid, lengths of parallel sides 1.6″ and 2.4″, distance apart 0.5″, one other side 0.7″.

***6.** Plot the quadrilateral ABCD to the dimensions given in the figure above. Measure the diagonal BD and the angle D. *Ans.* 4.58″, 86°.

7. Draw a *convex* irregular pentagon ABCDE, having given:—Sides AB = 2.91″, BC = 2.31″, CD = 3.56″, DE = 1.34″. Diagonals AC = 4.63″, BD = 3.98″. Angle DEA = 106.8°. Measure the remaining side and diagonals. *Ans.* EA = 3.04″, CE = 4.57″, DA = 3.66″, BE = 4.36″.

***8.** The survey of a plot of land is given in the figure, the dimensions being in feet. Draw the plot to a scale of $\frac{1}{8}$ inch to 1 foot.

N.B.—In 6, 7, and 8 verify the rule:—Number of dimensions $= 2n - 3$.

53. Enlarging and reducing polygons by the radial method.—We now give a fundamental construction, embodying a very important principle, by means of which a figure of any shape can be reproduced to any scale.

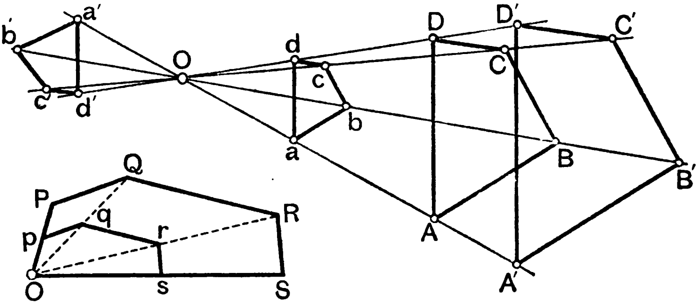
To illustrate, let the reader draw any polygon, say a quadrilateral ABCD. Mark any point O, and join OA, OB, OC, OD. Now starting from any point A′ in OA (or OA produced), draw in succession, as in the figure, A′B′ parallel to AB, B′C′ parallel to BC, C′D′ parallel to CD. Join D′A′, and *verify that* D′A′ *is parallel to* DA.

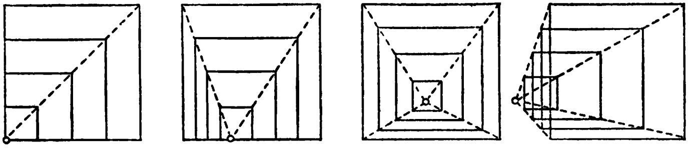
O is called the **pole**, the lines diverging from O are **radials**, and the construction as a whole is known as **radial projection**.

Let the student now further verify that the second quadrilateral is *similar* to the first. That is, that the angles A′, B′, C′, D′ are equal to the angles A, B, C, D, and that corresponding sides of the two figures are proportionals. In our illustration, using a $\frac{1}{8}''$ scale, OA measures 12 units, and OA′ 16 units. Therefore OA′ : OA = 16 : 12 = 4 : 3, or OA′ = 4/3 OA, and it will be found on measurement that OB′ = 4/3 OB, OC′ = 4/3 OC, OD′ = 4/3 OD, and A′B = 4/3 AB, B′C′ = 4/3 BC, C′D′ = 4/3 CD, D′A′ = 4/3 DA. In fact one quadrilateral, with its radials, is a reduced or magnified image of the other, just as if the shape of the larger figure were that of the smaller one when viewed through a lens which magnified one-third.

Two other reproductions to scale of the quadrilateral are shown at abcd, a′b′c′d′, the latter being obtained by extending the radials backwards beyond O. In the figure we have taken Oa′ equal to Oa, and these two quadrilaterals are congruent, that is, equal as well as similar. In order to understand their relation more clearly, let the figures be traced, and then inserting the pricker at O, let the tracing be turned through 180° and note what happens.

As a further example we show how a pentagon OPQRS may be copied to a linear scale of half size ($\frac{1}{4}$ the area), or how the pentagon Opqrs may be enlarged to a linear scale





of double size (4 times the area). The pole is taken at one corner O. The figure explains itself.

We have drawn the squares to show that in applying the radial method the pole may be taken in any convenient position inside, outside, or on the figure.

Examples.—**1.** Draw a square of 3″ side, and mark an internal point O distant 1″ and $\frac{1}{2}$″ from two sides. Join O to the corners, and draw three squares of 1″, 2″, and 4″ sides having their corners on the radials from O.

2. By using tracing-paper copy the above irregular pentagon OPQRSO. Also mark the line b′B′ of the upper figure. Then enlarge the pentagon, and alter its position such that the side OS shall be situated at b′B′.

3. Draw any trapezoid. Then draw a similar trapezoid with its longer side coinciding with the shorter one of the first figure.

4. Draw a regular pentagon (*a*) side 1.5″, (*b*) inscribed in a circle of 1.5″ radius.

5. By use of the table of chords, p. 25, (*a*) draw seven lines radiating from a point at equal angular intervals of 360° ÷ 7, (*b*) calculate the side of a regular heptagon inscribed in a circle of 2″ radius.

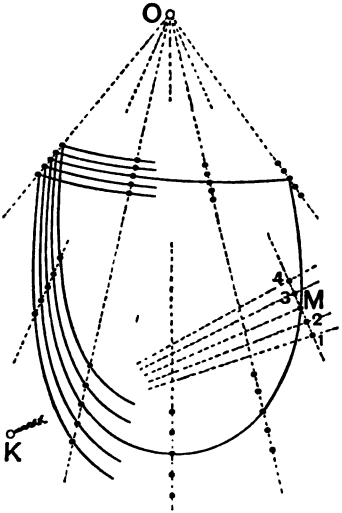
54. Enlarging and reducing curved figures. (a) By the radial method.—This method can be used for curved figures as well as for polygons. We find good illustrations in patterns of details of wearing apparel. To take a simple case, the figure shows the shape of the heel of a boot of a certain style and proportionate width. In grading this pattern to different sizes the shapes are similar. Having designed and drawn, say the middle size, the figure is readily "graded" by the radial method, as follows:—

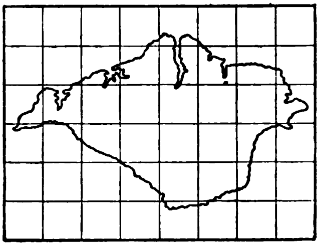
Through any pole O draw a number of radials. On tracing-paper draw the lines which are seen diverging from K. This is known technically as a radial grading tool. First set the tracing paper with K at O, and KM along any radial OM. Insert the pricker at M and turn the tracing through 90°. Then prick off the points 1, 2, 3, 4. This operation is to be repeated for a sufficient number of radials, so as to obtain consecutive points, through which fair curves can be sketched by freehand. These should be drawn with care, visible irregularities being corrected until, so far as can be judged from the appearance, the enlarged or reduced copies are exactly similar to the original. In an actual case there would be sizes intermediate between those shown.

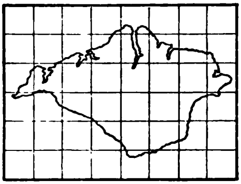
Let the student convince himself that the use of the radial grading tool KM as described, ensures the figures being similar. He has to satisfy himself that if a radial cut any pair of figures in say A, B, then the ratio OA : OB is constant for all radials.

(b) By the method of squares.—This is an alternative method of enlarging or reducing curved figures, illustrated by the two maps of the coast line of the Isle of Wight. Probably this device is already well known to the student, and its application is so obvious from the diagram as to make any written explanation superfluous.

55. On an important general method of solving problems.—As we have previously pointed out on p. 59, it often happens that from the nature of the data we can







readily draw a figure similar to the one required, but not of the right size. The method now referred to consists in drawing such a figure and then copying it to scale, or measuring its parts on another scale, or reducing them by arithmetic. Take the following simple illustration :—

PROBLEM.—*It is required to construct a right-angled triangle, hypotenuse 2″, ratio of sides 7 : 5.*

Solution.—Draw a right-angled triangle, the lengths of the sides being 7 and 5 on *any scale.* Then enlarge or reduce it so that its hypotenuse shall be 2″ long.

Examples.—**1.** Refer to p. 14. By the aid of tracing-paper, copy on your drawing paper, at sufficient distances apart, (*a*) the zig-zag line, (*b*) the two lower curves, and (*c*) the two lower plane figures. Then enlarge (*a*) and (*b*) to double size (linear), and enlarge (*c*) to three times the linear size. If possible adopt a different method of proceeding in each of the five cases.

2. Draw a sector of a circle in which the angle between the radii is 50° and the length of the arc 2.5″.

56. The locating of points. Systems of co-ordinates.—The student will have noticed that the defining and plotting, or the copying of figures resolves into the locating of points. There are three principal ways of doing this, each of which has already been employed. We now call formal attention to them, and it will be convenient to make use of a simple illustration.

Suppose a person wishes to note the position of a point in a plane, such for example as that of a small object like a boy's marble on the floor of a room, so that he may make a scale drawing which shall exhibit this position. After measuring the room he will require to make **two** further measurements of the position of the object. These two measurements are called the **co-ordinates** of the point.

In the figure P represents the point whose position in the room is to be recorded. It is evident from the figure that there is considerable latitude as to which two measurements or co-ordinates shall be chosen for the purpose. Three systems are in common use, and we shall confine attention to them. They are known as *rectangular* co-ordinates; *polar*, *radial*, or *vector* co-ordinates; and *triangular* co-ordinates or *triangulation*.

Rectangular co-ordinates.—The position of the object P in the room is known and can be plotted if we measure its distances from two adjacent sides. For example

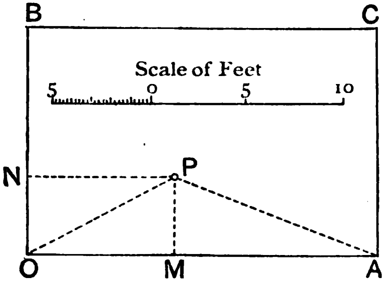
$$PN = 7.6 \text{ feet}; \quad PM = 4.0 \text{ feet}.$$

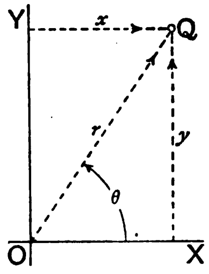
The method of defining form by measurements in perpendicular directions is an application of this system. See Chap. XIV.

Polar, radial, or vector co-ordinates.—A second method of locating P is to measure its distance from one corner O of the room, and the angle which the line OP makes with one side OA. Thus

$$OP = 8.6 \text{ feet}; \text{ angle } AOP = 28^\circ.$$

The radial figures of this chapter are based on this system. See also future chapters on vectors.





Triangular co-ordinates or triangulation.—Suppose P joined to two adjacent corners O, A of the room, thus forming a triangle OPA, of which the base OA is known. (It measures 18 feet.) The position of P might be determined by measuring any two other elements of the triangle OPA, thus enabling the triangle to be constructed. Say

Sides: OP = 8.6 feet; AP = 11.2 feet
or base angles, AOP = 28°; OAP = 21°.

If a second point P′ were required to be located, we might, if more convenient, take OP or AP as base, instead of keeping to the base OA. Proceeding in this manner, a series of points or the shape of any figure is determined by triangulation.

In the survey of the plot of ground, Ex. 8, p. 67, the shape of the main quadrilateral was found by triangulation, the remaining points being located by "*offsets*," which are rectangular co-ordinates.

In the method of surveying by the use of the plane table, see Art. 204, points are located by the intersections of lines whose directions are drawn. In this case the co-ordinates are the two base angles of the triangle.

Examples.—1. The right-hand figure above is drawn to a linear scale of half size. (*a*) OX and OY being the *axes of co-ordinates*, measure the rectangular co-ordinates of Q. *Ans.* $x = 1.48''$, $y = 2.23''$. (*b*) O being the pole, OX the reference line, and angles being estimated anti-clockwise, measure the polar or radial co-ordinates of Q. *Ans.* $r = 2.68''$, $\theta = 56.5°$.

2. If the rectangular co-ordinates of Q were $x = 1.5''$, $y = 1.2''$, find the radial co-ordinates. *Ans.* $r = 1.92''$, $\theta = 38.6°$.

3. Show that radial and rectangular co-ordinates of 2.5″, 37°, and 2″, 1.5″ correspond with one another.

57. Miscellaneous Examples.

*1. The figure gives the shape of the sole of a boot.

(*a*) Copy the figure accurately on tracing-paper, and enlarge it to double the linear dimensions.

(*b*) Alter the shape of your enlarged figure so that it shall be suitable for a wider, and also for a narrower, foot, dimensions parallel to the line from the heel to the toe remaining unchanged; and lateral dimensions, perpendicular to this line, being increased, and diminished, 10 per cent. Employ a "grading tool" drawn on tracing-paper.

*2. **The Pantograph.**—Let the sides, or sides produced, of any parallelogram be cut by any straight line in K, K, K, K. Suppose a model of this parallelogram to be made with hinged or jointed corners H, H, H, H. If the shape of the parallelogram is altered so as to move the points K nearer together or farther apart, it will be found that these points remain always in a straight line, and also that their distances apart remain in the same proportion.

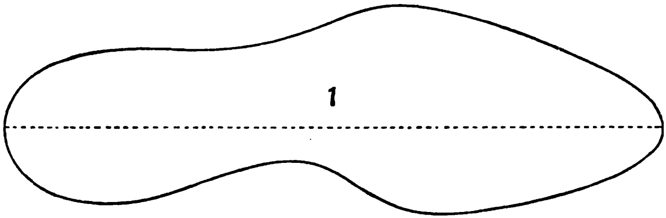
If, therefore, any one of the four points K be pinned to the paper at any pole O, and if a second point K trace out any figure, the two other points K will trace similar figures, larger or smaller according as to whether they are farther from or nearer to the pole.

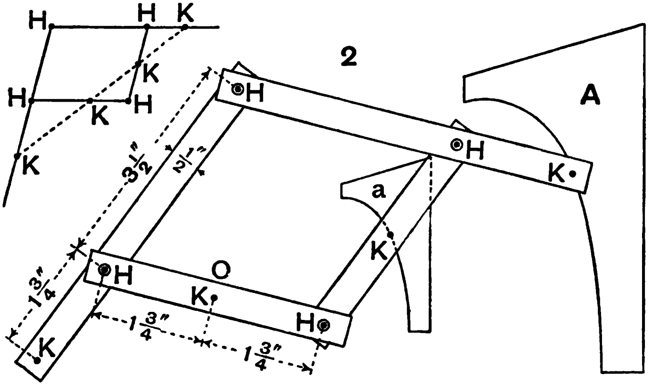
The dimensioned figure shows a useful size for the pantograph. It may be made of sheet celluloid of about the thickness of drawing-paper. The holes H, K may be pricked through with a needle, and the joints H made with fine soft iron wire passed through the holes and formed into flat scrolls on each side.

The diagram illustrates the instrument as it would be used when enlarging the figure **a** to double size, or reducing A to half size. It is pinned to the board with a needle at O. One of the points K is moved over the outline of the given figure, and at intervals points on its required copy are pricked through at the other point K.

It will be seen that by properly choosing three out of the four points K, figures can be enlarged in the linear ratios 2 : 3, 1 : 2 and 1 : 3, or correspondingly reduced. Other points K could be added for other ratios.

3. Draw any circle, centre C, say $\frac{3}{4}''$ radius. Take any pole O, say at a distance of $1''$ from C. Let P be any point on the circumference. Join OP, and produce OP to P′, making $OP' = 2.OP$. Do this for a number of points P round the circumference, and also for the centre C, obtaining C′. Observe the result. What is the path or locus of P′? What is the relation of C′ to this locus?





4. Copy the indicator diagram from p. 37. Then alter the size of the figure so that horizontal distances are doubled and vertical distances increased 50 per cent. What is the percentage increase of area? *Ans.* 200.

CHAPTER VIII

AREAS OF PLANE FIGURES

58 Area of the square and rectangle.—We have now to consider plane figures from a new standpoint, *viz.* as regards their **area**, or the amount of surface they contain.

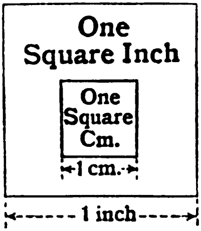
Just as lines were measured in terms of lines, and angles by comparison with other angles, so must the area of any surface be estimated by using some conveniently-selected area as a unit of measurement.

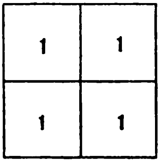
A square of 1″ side is shown; the amount of surface thus defined is called a **square inch.** Within it is drawn a square of 1 cm. side, its area being a **square centimetre.** Both these units are convenient for our purpose. The student may at once recall many others, as the acre, square mile, etc.

Now the area of any plane figure may be measured by finding how many of these unit squares would be required to cover it.

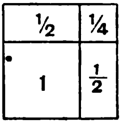
Thus a square of 2 cm. side can be divided into four squares each of 1 cm. side; its area is therefore 4 sq. cm. If the side were $\frac{1}{2}$ cm. the area would be $\frac{1}{4}$ sq. cm. And a square of $1\frac{1}{2}$ cm. side is seen from the figure to have an area of $2\frac{1}{4}$, that is $(1.5)^2$ square centimetres.

Again, from the figures it is evident that the area of a rectangle, sides 3 and 2 cm., is 3×2 or 6 sq. cm. And that of a rectangle, sides 2.4 and 1.3 **units of length,** is 2.4×1.3 or 3.12 **square units of area.**

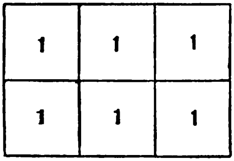




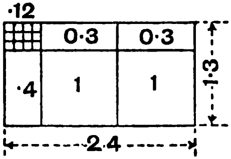
Side = 2 cm.
Area $= 2^2 = 4$ sq.cm.



Side $= 1\frac{1}{2}$ cm.
Area $= 1\frac{1}{2}^2 = 2\frac{1}{4}$ sq.cm.



Sides, 3 and 2 cms.
Area, = 3 x 2 = 6 sq. cm.



Sides, 2·4 and 1·3 units
Area, 2·4 x 1·3 = 3·12 sq. units.

Rules:- Area of Square = Side^2
Area of Rectangle = length x breadth.

Using further illustrations of his own, let the student convince himself of the general truth of the two rules given for the area of a square and rectangle.

Examples 1.—Draw a rectangle of any size, say about 3″ by 2″. Measure its sides accurately on the 1″ decimal scale, and also on the centimetre scale. Then calculate the area of the rectangle in square inches and also in square centimetres. By comparing the answers, find

(*a*) The number of sq. cm. in one sq. inch. *Ans.* 6.451.
(*b*) The number of sq. inches in one sq. cm. *Ans.* 0.1550.

2. Show by means of a figure, drawn to scale, that the area of a square of 1.6″ side is 2.56 sq. inches. Also that the area of a rectangle, sides 7.8 and 5.4 cm. is 42.12 sq. cm.

3. A rectangular lawn 25.3 yards long by 15.7 yards wide is bounded by a 5 feet path. Draw the plan of path and lawn to a scale of $\frac{1}{8}$ inch to 1 yard. Find the area of the path. *Ans.* 147.8 square yards.

59. Area of the parallelogram.—Let AABC be any parallelogram, and let BD be perpendicular to AA. Then

area of parallelogram = AA × BD = **base × altitude.**

In verification, a model of the parallelogram might be made in paper, and the corner ABD cut off and shifted to ACD. Thus, without altering its area, its shape would be changed to a rectangle of equal base DD and the same altitude. And the area of the rectangle is known. Hence the above rule.

It is thus evident that *parallelograms which have equal bases, and equal altitudes, have also equal areas.*

Note.—We have BD ÷ BA = sin A; or BD = BA sin A. Therefore area of parallelogram = AA × BD = AA × BA sin A. That is

area = product of sides × sine of included angle.

60. Area of the triangle.—Let EEF be any triangle and let FG be perpendicular to EE. Then the rule is

area of triangle = $\frac{1}{2}$ EE × FG = half base × altitude.

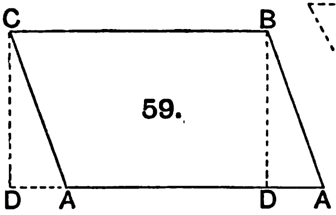
Two ways of verifying this rule are suggested in the figure. In one it is shown how a model might be cut so as to be able to transfer R to R′ and S to S′, thus reducing the triangle to an equivalent rectangle on the same base and of half the altitude. In the other a duplicate triangle is to be made and placed as shown at T, thus making up a parallelogram of which the triangle FEE forms one half.

Let two triangles have equal bases HH, H′H′ in the same line, and have their vertices K, K′ in a parallel line. Their altitudes and areas are evidently equal. Hence the important proposition.

Triangles on the same base or on equal bases, and between the same or equidistant parallels are equal in area.

Note.—We have FG ÷ FE = sin E; or FG = FE sin E. Therefore area of triangle = $\frac{1}{2}$ EE × FG = $\frac{1}{2}$ EE × EF sin E. That is *area = half product of sides × sine of included angle*

61. Area of the trapezoid.—Let LL, MM be the parallel sides of the trapezoid, and PP the perpendicular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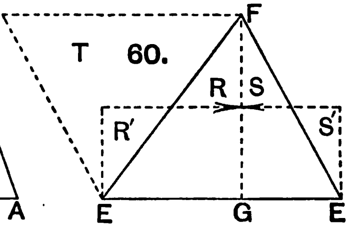
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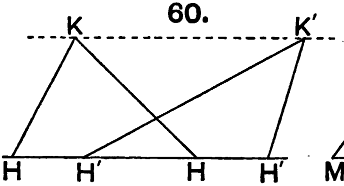
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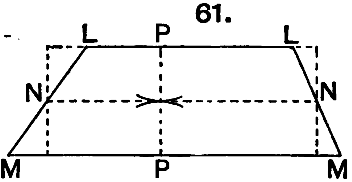
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distance between them. Draw **N N** parallel to the parallel sides and midway between them. Then

area of trapezoid = N N × P P = **mean length** × **breadth.**

The truth of this rule should be evident from the figure, the dotted lines indicating how the trapezoid could be reduced to an equivalent rectangle. Let the student verify that N N is half the sum, or the mean of L L and M M.

Examples. — **1.** Draw an equilateral triangle of 3.5″ side. Measure its altitude. Find its area in square inches and square centimetres. *Ans.* Altitude = 3.03″, area = 5.31 sq. in. or 34.2 sq. cm.

2. Draw a parallelogram, sides 11 and 5 cm., included angle 50°. Find its area in square centimetres and in square inches. *Ans.* 42.13 sq. cm.; 6.53 sq. in.

3. Draw a triangle, sides 12 and 6.5 cm., included angle 70°. Find its area in square centimetres and in square inches. *Ans.* 36.6 sq. cm.; 5.68 sq. in.

4. Construct a triangle, base B B, 3.4″; sides B A, 2.7″ and 2.1″. Draw a line C C parallel to the base so as to bisect the sides. Find the areas of the triangles A B B, A C C, and of the trapezoid B B C C. Find the ratios of these areas to one another.

62. Area of any quadrilateral.—If one of the diagonals AA be drawn, the quadrilateral is divided into two triangles, having a common base AA, and altitudes BD, CD, see the figure. The area of each triangle can then be found and the two areas added. Thus

$$\text{area} = \tfrac{1}{2} AA \times BD + \tfrac{1}{2} AA \times CD = \tfrac{1}{2} AA(BD + CD).$$

This suggests how the quadrilateral can be reduced to an equivalent rectangle.

An **alternative** plan is to first reduce the quadrilateral to an *equivalent triangle*, and then to find the area of the latter.

To effect this, through one corner F (see the right-hand figure) draw a line parallel to the diagonal EE, meeting either of the sides GE produced in H. Join EH. Then either of the triangles EGH is equal to the quadrilateral EFEG.

In explanation, let the student recognise that the triangles EHE, EFE, EHE are all equal to one another, because they have the same base EE and equal altitudes (being between the same parallels).

63. Area of any polygon.—If lines be drawn from one corner to all the other corners, the polygon will be divided into triangles, the areas of which can be found separately, and the results added.

A better plan will probably be to reduce the polygon to an equivalent triangle, by an extension of the second method just employed in the case of the quadrilateral.

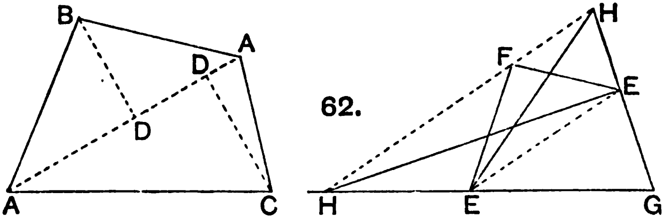
The construction for effecting this is shown in the figure. We leave the student to make this out for himself. Note that Lm, mn, np are parallel to KM, KN, KP, and Ts, sr, rq parallel to KS, KR, KQ.

Examples.—1. Draw a quadrilateral PQRS having given :—sides, PQ = 2″, QR = 1.9″, RS = 1.1″, SP = 1.6″; diagonal PR = 2.2″. Reduce this figure—

(*a*) To an equivalent rectangle on the diagonal PR.

(*b*) To an equivalent triangle, vertex at P, base along QR.

Find the area of the quadrilateral. *Ans.* 2.61 sq. in.



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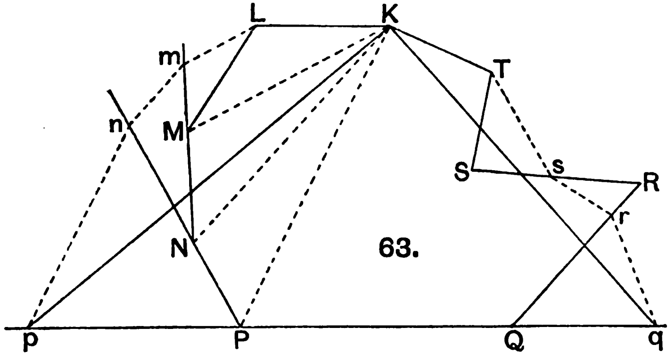


63.

***2.** Copy the polygon KLMPQRST, Fig. 63, and enlarge it to double the linear size. Then reduce the polygon to a triangle with vertex at L and base along PQ. Find the area of the enlarged polygon. *Ans.* 10.1 sq. in.

3. If the trapezoid LLMM in the figure, p. 79, be drawn to a linear scale of $\frac{1}{4}$, what is the area represented by it? *Ans.* 12.9 sq. in.

4. The quadrilateral ABAC, Fig. 62, is drawn to a scale of 1 cm. to 1 inch. Copy the figure *full size*. Then find its area in square inches and also in square centimetres. *Ans.* 5.66 sq. in. 36.5 sq. cm.

5. Find the areas of a parallelogram and triangle, sides 4″ and 2.5″, included angle 50°. Find also the area of a quadrilateral, diagonals 4″ and 2.5″ included angle 50°. *Ans.* 7.66, 3.83, 3.83 sq. in.

64. Further examples of equivalent figures.—We have shown how to reduce certain simple figures to others of equal area. We now give some additional cases which are of interest.

65. PROBLEM.—**To reduce a given triangle ABC to an equivalent triangle on the base BD.**

Join DA. Draw CE parallel to DA. Join ED. Then EBD is the required triangle.

For the triangles ECA and ECD are equal, being on the same base EC and between the same parallels EC, AD.

66. PROBLEM.—**To reduce a given rectangle AC to an equivalent rectangle on the base BD.**

Join DA. Draw CE parallel to DA. Draw the rectangle ED, which is the one required.

We have used the same letters as before, in order to make more prominent the similarity in the two constructions. Indeed the proof is the same, observing that the rectangles AC, ED are double the triangles ACB, EDB. The following is an alternative proof:—

The triangles EBC, ABD being similar, we have

$$EB : BC :: AB : BD; \text{ or } EB \times BD = AB \times BC,$$

which proves the equality of the areas of the two rectangles. A similar construction is shown, applied to a parallelogram.

67. PROBLEM.—**To reduce a given rectangle FGH to a square of equal area.**

Produce one side HG to K, making GK equal to the other side GF. Bisect HK in O, and with centre O describe the semicircle on HK. Produce (if necessary) GF to L. On the side GL construct the required square LM.

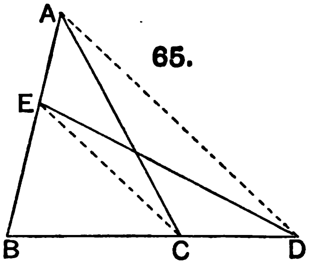
The student may test the accuracy of this construction in any particular case by actual drawing, measurement, and calculation. The reason will appear in the next chapter. See Arts. 100 and 101.

It will there be shown that GL is a *mean proportional* between GK and GH. That is

$$GK : GL :: GL : GH$$

or

$$GL^2 = GH \times GK = GH \times GF.$$



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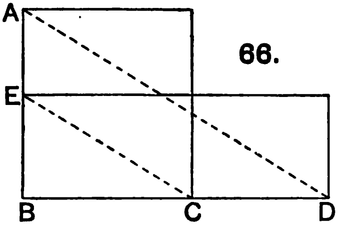

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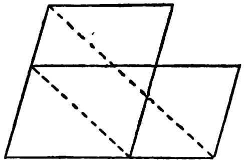
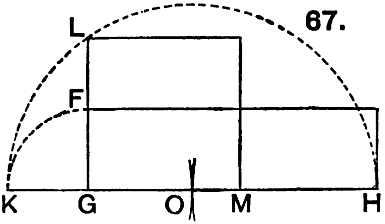




67.

Examples.—**1.** Draw a triangle, base 2.5″, sides 2.7″ and 1.9″. Reduce the triangle to equivalent triangles as follows :—

(*a*) To an isosceles triangle on the same base.
(*b*) To an isosceles triangle on a 3″ base.
(*c*) To an isosceles triangle having an altitude of 2.3″.
(*d*) To an isosceles triangle having the same vertical angle.
(*e*) To a right-angled isosceles triangle.

2. Reduce the triangle of Example 1 to a square of equal area.

3. Draw a rectangle, sides 2.9″ and 1.7″. Reduce the figure to a rectangle, one side of which measures 2.3″.

4. Draw a parallelogram, one side 3″ long, included angle 55°, and equal in area to an equilateral triangle of 2.5″ side.

5. Draw a square of 2″ side. Enlarge the figure to double the linear dimensions. What is the increase of area? What is the percentage increase of area? *Ans.* 12 sq. in.; 300 per cent.

6. A square field has a side 50 yards long. Draw its plan to a scale of $\frac{1}{2}$″ to 10 yards. Draw a circle having the same area and centre as the field. If a goat is tethered by a rope at the centre, find the lengths of the rope in order that the animal may command $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ the area of the field.

68. Some general propositions relating to areas.—In the diagram opposite are shown three triangles, all of the same altitude a, but having bases of different lengths b_1, b_2, b_3. Their areas are $\frac{1}{2}ab_1$, $\frac{1}{2}ab_2$, $\frac{1}{2}ab_3$.

Three other triangles are shown with equal bases b, and unequal altitudes a_1, a_2, a_3. The three areas are $\frac{1}{2}a_1b$, $\frac{1}{2}a_2b$, $\frac{1}{2}a_3b$.

These expressions for the areas are equivalent to saying: *In triangles of equal altitudes the areas are proportional to the bases.* And, *in triangles with equal bases the areas are proportional to the altitudes.*

If both altitude and base are different, the area is proportional to the *product* of the two.

Let us now consider triangles which are **similar** in shape but of different size. In this case it will be found that the areas are proportional to the *squares* of the bases, or the altitudes, or more generally, to the *squares of their linear dimensions.*

Thus an equilateral triangle is shown, divided up into four smaller similar triangles. Each of the latter is *half* the linear size of the former, and has one *quarter* the area.

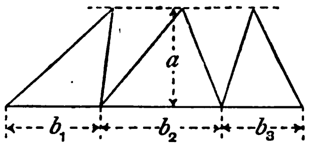
Or, again, let ADD be any triangle. Trisect its sides and draw the parallel lines as shown, thus dividing ADD into nine triangles all equal. Now observe that the triangles ABB, ACC, ADD are similar; their linear dimensions are in ratios 1 : 2 : 3; their areas in the ratios 1 : 4 : 9, *i.e.* $1^2 : 2^2 : 3^2$.

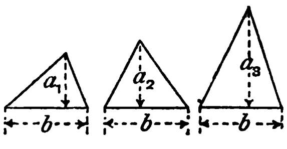
As an additional example, two similar quadrilaterals, EF, EG are shown. One is of *twice* the dimensions of the other, and has evidently *four* times the area.

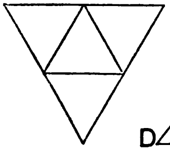
And if a plane figure of any shape be magnified in linear dimensions n (say 20) times, its area is magnified n^2 (or 400) times. Thus generally:—

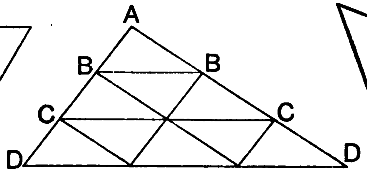
The areas of similar figures are proportional to the squares of their linear dimensions.

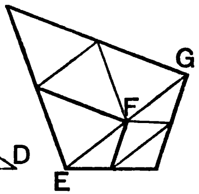
The Theorem of Pythagoras (Euc. I. 47), illustrated in Art. 41 (*b*), is true not only for squares, but also for any

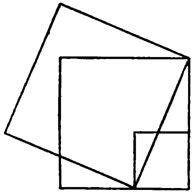


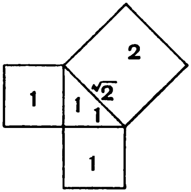


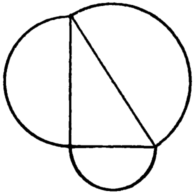












other similar figures. *E.g.*, semicircles are such figures, and we therefore have the proposition :—

The semicircle described on the hypotenuse of a right-angled triangle is equal to the sum of the semicircles described on the sides.

The lower middle figure indicates how to draw a square of double, or half, the area of another square. Instead of squares, we might have circles, equilateral triangles, or any figures which are of the same shape.

The diagrams also suggest how to draw a square, or figure, equal to the *sum*, or *difference*, of two other squares, or corresponding similar figures.

Example.—Draw a quadrilateral similar to and double the area of the quadrilateral EG above.

69. Areas of curved figures.—We now explain how to find, approximately, the area of any figure having a curved outline. The method consists in reducing the figure to one or more equivalent rectangles.

Consider first the figure ABBBA, which has a straight base AA, two parallel sides AB at right angles to the base, and a curved side BBB.

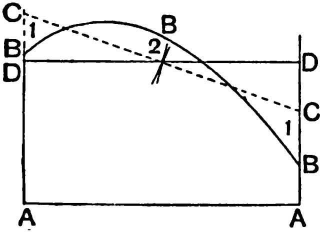
If only a rough approximation were required, we might first draw a straight line on tracing-paper, superpose this on the figure, and adjust it to some such position as CC, until, *to the best of our judgment*, the areas marked 1, 1 below CC were together equal to the area marked 2 above it. The trapezoid ACCA would, with more or less error, be equal in area to the given figure. The trapezoid could then be reduced to the rectangle ADDA, as in Art. 61, and the area of the rectangle found by calculation.

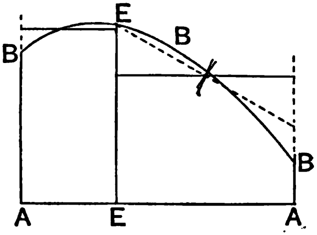
To ensure a better approximation, the figure might be divided into two or more parts, by a line or lines like EE, perpendicular to AA. Then, proceeding as before, reduce each part to an equivalent rectangle. Calculate the areas of the rectangles and add.

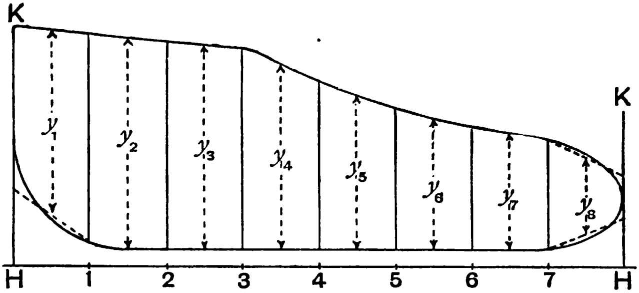
It is usual, however, to proceed in a more systematic manner, and, in illustrating this, we will take the lower figure, in which the outline is entirely curved.

Draw two convenient parallel lines like HK, HK, just touching the curve and enclosing it between them. And draw a line HH perpendicular to HK. Now consider HH as a base, and let us reduce the curved figure to an equivalent rectangle on this base.

Divide HH into a number of equal parts, usually from 6 to 12, according to the degree of accuracy required; there are 8 in the diagram. Erect perpendiculars at the points of division as shown. The curved figure is thus divided into a number of strips, or trapezoids with curved ends, all of equal width. We must next find the lengths, or heights, y_1, y_2, y_3 etc. of these strips, at the middle of their widths. These may be called the **mid-ordinates** of the strips.







The area of any strip is very approximately equal to its mid-ordinate multiplied by its width. In our case the width of any strip is $\frac{1}{8}$th the base HH. The total area of all the strips, that is of the figure, is thus equal to $y_1 \times \frac{1}{8}\text{HH} + y_2 \times \frac{1}{8}\text{HH} + \ldots.$ That is

$$\text{Area} = \frac{1}{8}(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8) \times \text{HH}.$$

The sum of all the ordinates, divided by their number is called the **mean ordinate.** The area required is thus the mean ordinate multiplied by the base.

Thus the mean ordinate is the average height of the curved figure, and it gives us the height of the equivalent rectangle on base HH. This rectangle is omitted, in order not to overload the figure.

70. The area of the circle.—We might proceed as in the general case, and split up the circle into parallel strips. In this case, however, it is better to adopt a special method, and to divide it up into a number of equal sectors as shown for the semicircle. Now suppose these sectors to be separated and rearranged as indicated on the right.

From this figure it at once becomes evident, that if the sectors were taken smaller and smaller, their number being correspondingly increased, the re-grouped figure would eventually become a rectangle, its base being equal to the semi-circumference, and its height equal to the radius.

The answer therefore is at once apparent, viz. :—

Area of circle = half circumference × half diameter.

We have previously found, see Art. 18, Ex. 2, that the circumference of any circle is 3.142 or π times the diameter. So that we have the following important formula :—

$$\textit{Area of circle} = \tfrac{1}{2}\pi d \times \tfrac{1}{2}d = \frac{\pi}{4}d^2 \text{ or } = \pi r^2$$

where d stands for diameter, r for radius, and π for 3.142.

Area of a sector.—It is evident that for sectors of circles of the same radius, the areas are proportional to the angles they contain. Thus if the angle were 60°, the area would be 60 ÷ 360ths or $\frac{1}{6}$th of the area of the full circle.

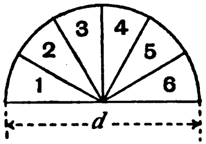
71. Summary of chapter.—The student must acquire an intimate knowledge of the following propositions :—

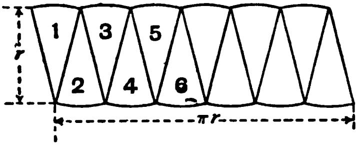
Theorem 1.—*Parallelograms are equal in area,* **(a)** *when they are on the same base or on equal bases, and between the same or equidistant parallels.* **(b)** *When the product base × altitude is the same.*

Theorem 2.—*Triangles are equal in area under like conditions.*

Theorem 3.—*The area of a parallelogram is equal to that of a rectangle having the same base and altitude.*

Theorem 4.—*The area of a triangle is equal to half that of a parallelogram which has the same base and altitude.*





Theorem 5.—*The areas of parallelograms, or of triangles, on equal bases, are proportional to their altitudes.*

Theorem 6.—*The areas of parallelograms, or of triangles, of equal altitudes, are proportional to their bases.*

Theorem 7.—*The areas of similar figures are proportional to the squares of their linear dimensions.*

Rule 1.—*Area of square* $= (\textit{side})^2$.

Rule 2.—*Area of rectangle = length × breadth.*

Rule 3.—*Area of parallelogram = base × altitude.*

Rule 4.—*Area of parallelogram = product of sides × sine of included angle.*

Rule 5.—*Area of triangle = half base × altitude.*

Rule 6.—*Area of triangle = half product of two sides × sine of included angle.*

Rule 7.—*Area of trapezoid = width × mean length.*

Rule 8.—*Area of circle = semi-circumference × radius =*

$$\frac{\pi}{4}d^2 = \pi r^2. \quad [\pi = 3.142.]$$

Examples.—**1.** Draw a circle of 2″ radius. Find the length of its semi-circumference by the use of tracing-paper. Draw a rectangle equal in area to the circle. Reduce the rectangle to a square. Set out the circle and equivalent square to the same centre. What is the area of the circle? *Ans.* 12.57 sq. in.

2. Draw a circle of 2″ radius. Divide it up into three sectors having their areas in the ratios 2 : 3 : 4.

Hint.—The angles of the sectors have these ratios.

3. Draw a circle of 2″ radius. Divide it up by concentric circles into parts having their areas in the ratios 2 : 3 : 4.

4. Draw three circles, the areas of which shall be equal to the areas of the sectors in Ex. 2.

5. Two sides of a triangle measure 3″ and 2″ and the angle included between them is 45°. Calculate the area by Rule 6. Confirm by Rule 5.

72. Miscellaneous Examples.

***1.** The triangle **abc** is given. Construct an isosceles triangle of equal area, standing on **de** as base. (1889)

***2.** Reduce the figure **abcd** to a triangle with **A B** as base, and the vertex in **bc**. (1891)

***3.** Divide the triangle **A B C** into four equal parts by lines drawn parallel to **A B**. (1896)

***4.** Find the area of the given hatched figure. (It is drawn half the linear size on the diagram.) (1883)

***5.** Reduce the figure to a rectangle on the base **B B**. Find the area of the figure in square centimetres (after being enlarged to double the linear size). *Ans.* 40.0 sq. cm.

6. Find, in sq. cm., the area of the segment drawn on p. 93. *Ans.* 2.07 sq. cm.

7. Find the area of the sector, and of the quadrant, drawn on p. 93. Find the ratio of the two areas. *Ans.* Ratio = 0.643.

8. The survey of a plot of ground is given in Fig. 8, p. 67, the dimensions being in feet. Draw the plot to a scale of $\frac{1}{8}$ inch to 1 foot. Find the area of the plot in square feet, and also in square yards. *Ans.* 493 sq. ft., 54.8 sq. yds.

9. Copy the figure **A B B B A**, p. 87, then enlarge it to twice the linear size. Now find its area (*a*) as shown by the left-hand figure; (*b*) as indicated in the right-hand figure; and (*c*) by the method illustrated in the lower figure, the base **A A** being divided into six equal divisions. Compare the three answers and note the percentage difference of each from the average value of the three.

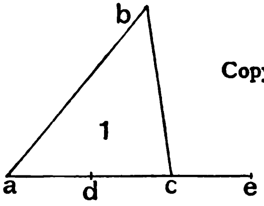
10. Copy the lower figure, p. 87, then enlarge it to double the linear dimensions. Now find its area, by the method explained in Art. 69, dividing the base **H H** into ten equal parts instead of eight as shown in the figure. Find the mean ordinate, and compare your answer with the mean ordinate as obtained from measurements of the eight mid-ordinates directly from the figure on p. 87.

11. Draw a semicircle, the area of which shall be 3.2 square inches.

12. Draw a semicircle, the area of which shall be equal to the sum of the areas of two semicircles of $1\frac{1}{2}''$ and $2''$ diameter. Measure its diameter. *Ans.* $2\frac{1}{2}''$.

13. Draw an equilateral triangle which shall be equal to (*a*) the sum, (*b*) the difference of two equilateral triangles of $4''$ and $2''$ sides. Measure their sides. *Ans.* (*a*) $4.47''$, (*b*) $3.48''$.

14. Calculate the area of a circle of $2.2''$ radius. Calculate the area of a sector of this circle, the angle between the radii being 100°. Find the area of the triangle formed by the two radii and the chord joining the ends of the arc of the sector, and deduce the area of the remaining segment.



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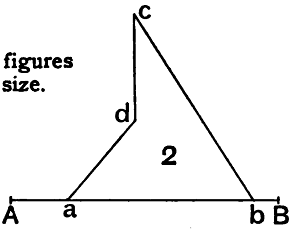
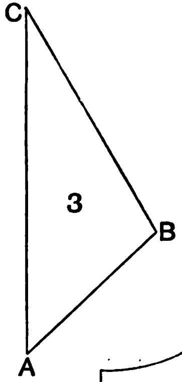
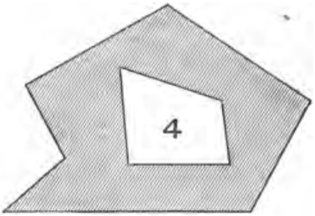
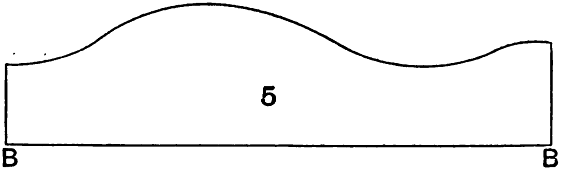








Note.—It should be observed that in the class for Practical Mathematics, which the student will probably be attending, the rules for areas, along with other mensuration rules, will be illustrated and verified in a number of different ways, such as by drawing the various figures on squared paper, and counting the number of squares enclosed; or by cutting out their shapes in thin sheet material, and then comparing their weights as ascertained by a balance; etc. Some teachers may adopt these methods in the drawing class in order to illuminate the subject.

CHAPTER IX

THE CIRCLE

73. Explanation of terms.—We first illustrate and give the meanings to be attached to some common terms.

A **circle** is a plane figure bounded by a curve, all points of which are equidistant from a point, called the **centre.**

The **circumference** of a circle is the bounding curve.

An **arc** of a circle is any portion of the circumference.

A **radius** is a line from the centre to the circumference.

A **chord** is a straight line joining two points in the circumference.

A **diameter** is a chord which passes through the centre.

A **segment** is a portion of a circle bounded by an arc and chord.

The **angle in a segment** is the angle between two lines which join any point in the arc to the ends of the chord.

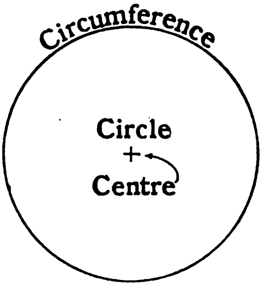
A **sector** is a portion of a circle bounded by an arc and two radii.

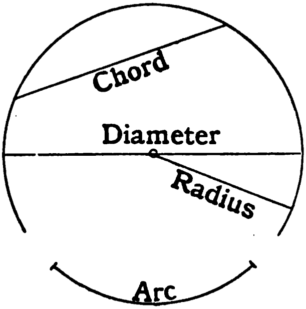
A **quadrant** is a sector in which the radii contain 90°.

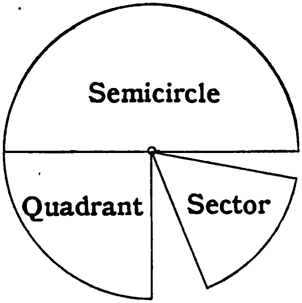
Concentric circles are such as have the same centre.

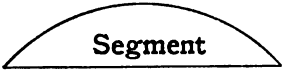
A **tangent** to a circle is an extended chord passing through two indefinitely near or consecutive points on the circumference. See Art. 74.

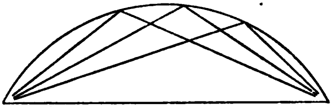
Circles are **tangential**, or are said to **touch** one another, when they have two consecutive points on each circumference in common. See Art. 78. The contact may be **external** or **internal.**



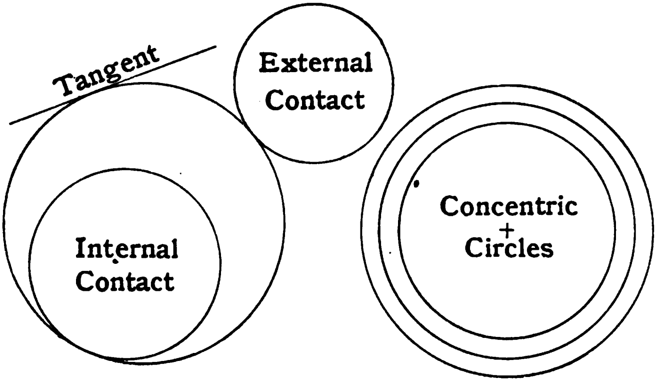








Angles in Segment.



74. Some fundamental properties verified.—The following properties of the circle should be verified experimentally by accurate drawing and measurement of lengths and angles. But first let the student see to his compass lead; it should be fairly hard, say H or HH, and must be carefully sharpened, first with a knife, and then finished on glass paper to a long fine chisel edge. Also let the drawing pencil be put in good order.

Construction.—Draw any circle and mark its centre A. Then draw any radius AB. And draw any chord CC perpendicular to the radius. Now verify that—

Theorem (a).—*Any chord of a circle is bisected by the perpendicular radius.*

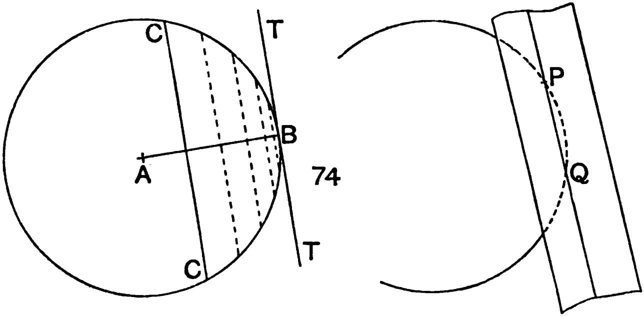
Construction continued.—Draw chords parallel to CC, farther and farther from the centre. Observe that these chords get shorter and shorter the nearer they are to B. At the instant before B is reached, the points CC are indefinitely near together (we may call them *consecutive points*), and in the limit, when B is actually reached, the chord CC, produced both ways, becomes the *tangent* TT *at* B. The student has thus verified the following very important theorem :—

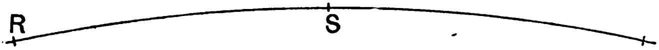
Theorem (b).—*The tangent at any point* B *on the circumference is perpendicular to the radius from* B.

Experiment.—Draw a circle (in ink by preference). Place over it a strip of tracing paper on which a straight line has been drawn, cutting the circumference in two points, say P and Q. Insert the pricker at P, and rotate the line round P in the direction which brings Q towards P, and continue the movement until, *so far as can be judged,* Q has come up to P, and the line has become the tangent at P. Prick off and draw this tangent.

This process is given for the purpose of illustrating the meaning of a tangent. It is not recommended as a practical way of finding the latter; the method should rather be avoided as being ill-conditioned. The student may draw the radius from P, and test whether his tangent is actually perpendicular to it; he will probably find an error of several degrees. The proper way of proceeding is given in Prob. 75.

From an *external point,* a tangent may be drawn without any previous construction; but in order to locate the point of contact accurately, apply Prob. 76.





75. PROBLEM.—**To draw the tangent at any point P on the circumference of a circle.**

First draw the radius from P. Then draw the required tangent perpendicular to this radius.

76. PROBLEM.—**Having drawn a tangent to a circle, to find the point of contact.**

Draw a radius perpendicular to the tangent, intersecting the latter in the required point.

77. PROBLEM.—**To draw a circle of given radius a, to touch a given line T T at a given point B.**

Draw BA perpendicular to TT, making BA equal to a. Then A is the centre of the required circle.

Examples.—**1.** Draw any circle and obliterate its centre. Invent several ways of finding the centre.

***2.** Invent a method of drawing accurately a tangent at S to the flat circular arc given above. Draw also a **normal** to the curve at S, that is, a portion of the radius.

3. Verify experimentally the converse of Theorem (a), Art. 74, viz. :—*The perpendicular bisector of any chord of a circle passes through the centre.*

4. Draw a circle 1.5″ radius. Mark a point P 2″ from its centre. Draw the tangent from P and find the point of contact M. Measure PM. *Ans.* 1.32″.

78. Properties of a pair of circles.—On any straight line mark two points AA, and with these as centres, describe intersecting circles, and draw the common chord CC. (Two cases are shown in the figure.) Verify that AA bisects CC at right angles. Also that the arcs CC on both circles are bisected by AA. Thus:—

Theorem (a).—*If two circles intersect, the common chord is bisected at right angles by the line joining their centres.*

Continuing, let AA cut one of the circles in B. (Two cases are still shown.) Draw circles concentric with the other circle, altering the radius each time so that the chord CC gets shorter and shorter, and ultimately the points C,C become consecutive at B, and common to the two circles and the chord. Thus in the limit the circles touch one another at B, and the extended common chord becomes the common tangent TT at B. The student thus verifies that:—

Theorem (b).—*If two circles touch, the point of contact is in the line through their centres. And they have a common tangent at this point, which is perpendicular to the line of centres.*

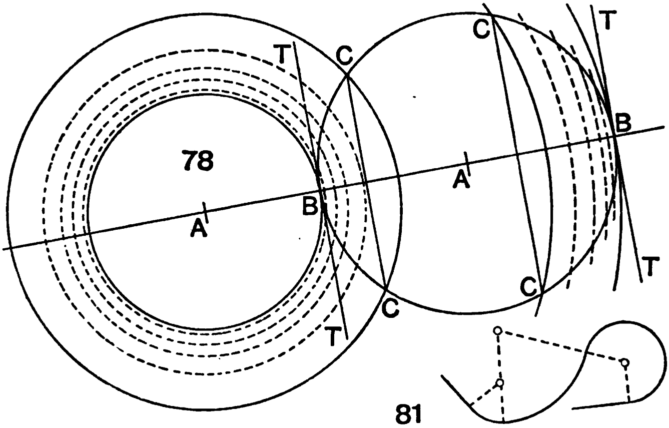
The two cases shown are for internal and external contact. The properties are identical in both.

Remarks.—The student may experiment with tracing paper in the manner explained in Art. 74, but taking a circle instead of a straight line on the tracing. He should devise tests of his own, and acquire information at first hand. For instance, if he takes different tracings, with circular arcs on them getting flatter and flatter, he may perhaps realise that the previous case of Art. 74 is only a particular case of the present more general one, when the radius of one of the circles increases without limit.

The student should now be capable of applying the properties he has just learned to problems like the following:—

79. PROBLEM.—**To find the point of contact of two tangential circles, or arcs of circles.**

Join their centres, intersecting the arcs in the required point.



80. PROBLEM.—**To draw a circle of given radius, a, to touch a given circle at a given point B.**

Draw a radius through B, on which mark off BA equal to the given radius *a*. With centre A describe the required circle.

Note.—There are two possible solutions, corresponding to external and internal contact.

81. Lines and arcs joining tangentially.—Cases arise in which we require to find the junction of two circular arcs, or of a line and an arc, which meet tangentially, *i.e.* which have a common tangent at the join. We wish to find the point where one arc or line ends and the other begins. This is a simple application of the principles already established, and the small figure above should be sufficient illustration.

Examples.—1. Find the centre of any circle by using merely a straight edge with two parallel edges, and a pencil, and without employing compasses.

2. Draw a series of tangential arcs and lines, suitable centres and radii being chosen at pleasure. Then find the junctions of the several parts of the figure.

3. Draw a circle. Mark any point as centre and draw a tangential circle. Then find the point of contact.

82. The notion of a locus.—The next chapter is specially devoted to loci, but a preliminary idea is helpful at this stage. From the present point of view any line, straight or curved, is conceived as having been traced by a point moving under suitable conditions. The path of the moving point, that is the curve generated by it, is called the **locus** of the point.

Thus in using a straight edge and drawing pencil, the path or locus of the pencil point, moving under the constraint of the human muscles, the edge of the ruler, and the paper is a straight line. If a parallel line be drawn, this may be regarded as the locus of a point which is always at a constant distance from the first line. Or again, when a circle is described, the student may think of the curve as the locus of a point which remains at the same distance from a fixed point. The locus of a second point which moved while remaining at a constant distance from the circle would be a concentric circle.

Mark two points A, A. With these as centres, any radius, describe intersecting arcs. They cut in a point which is at equal distances from A, A. See the figure. We can repeat this process time after time with different radii, and we can imagine it to be repeated indefinitely, so as to obtain all possible points having this property. It is clear from the figure that the aggregate of the points would constitute the perpendicular bisector of AA. Thus the perpendicular bisector of a line may be regarded as the locus of a point which is always equidistant from the ends of the line.

Take a third point A, associate it with either of the first two points A, and draw the perpendicular bisector of the new line AA. We get a second locus similar in character to the first.

It will be seen at once that the intersection of the two loci gives us a point O which is at equal distances from all three points A. A circle, centre O, passing through one of the points A, would, if the construction were accurately made, also pass through the other two points A.

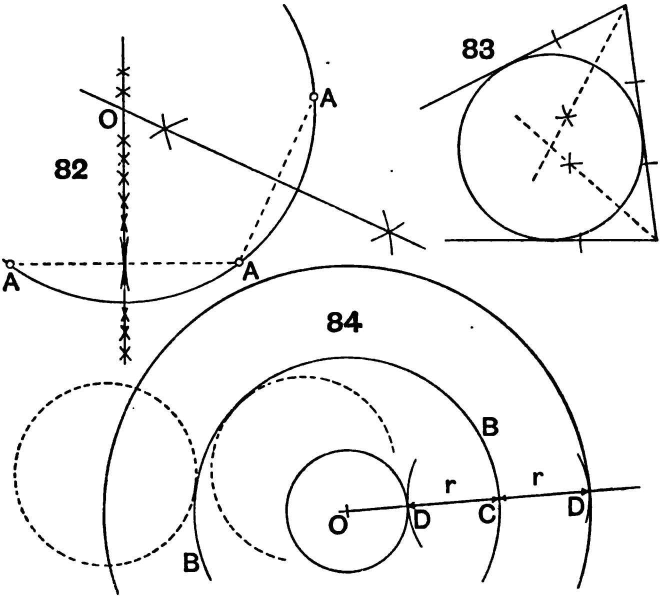
This construction would solve the following three problems :—

(a) PROBLEM.—**Describe a circle which shall pass through three given points A, A, A.**

(b) PROBLEM.—**Draw the circle which circumscribes a given triangle AAA.**

(c) PROBLEM.—**Find the centre of any given circle, or circular arc AAA.**

On the next three pages we give further problems readily solved by the aid of loci.



83. Problem.—**To draw a circle which shall touch three given lines, or which shall be inscribed in a triangle.**

See the figure, and read the above remarks on loci.

84. Problem.—**To find the locus of the centre of a circle, radius r, which always touches a given circle BB.**

Draw and produce any radius OC, on which mark off OD equal to *r*. The required locus is the concentric circle through D.

Note.—There are seen to be two solutions, corresponding to external and internal contact.

Example.—Draw the locus of a point which moves so as to be (*a*) always 2″ distant from a fixed point; (*b*) always 2″ distant from a fixed line; and (*c*) always 2″ distant from a fixed circle of radius 0.5″. *Ans.* (*c*) Concentric circles, radii 2.5″ and 1.5″.

85. The locating of circles of given radius.—The problems which we have brought together on this page have one common feature, the object in all being to find the position of a circle or circles whose size is known.

In the figures the results are arrived at by the plotting of loci in which the centres of the required circles must lie; the intersections of the loci give us the centres. For the sake of clearness the loci are indicated by dotted lines, and their intersections are all marked O. If a student has read the two preceding pages the solutions will be obvious from the figures, so that a written description in each case is unnecessary.

Application of tracing-paper.—An alternative and interesting way of dealing with these problems is: First draw a circle of the required size on tracing-paper, then adjust this in position over the datum lines drawn on paper, and prick off the centre. The student is recommended to try this method.

86. PROBLEM.—**To describe a circle of given radius r to pass through two points A, A.**

From the figure it appears that the loci intersect in two points O, O, giving two solutions. Only one of the circles through A A is drawn.

87. To draw a circle of given radius r to pass through a point B and touch a given circle C C.

The number of solutions may vary from nothing to four, according to the data.

88. To draw a circle of given radius r to touch two lines D D, D D.

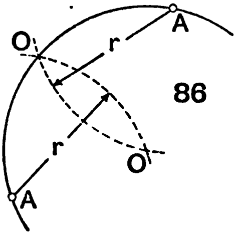
There are four solutions.

89. To draw a circle of given radius r, to touch a line E E, and a circle F F.

The number of possible solutions ranges from 0 to 8.

90. To draw a circle of given radius to touch two given circles.

Let the student himself arrange the data, draw the figure, and find the maximum number of solutions.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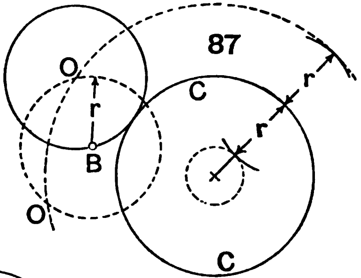
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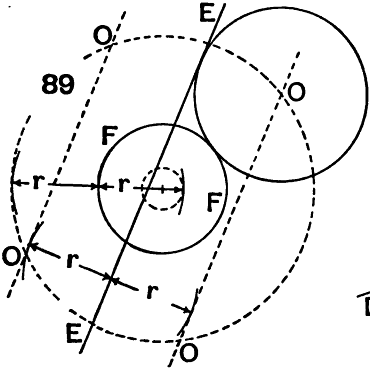
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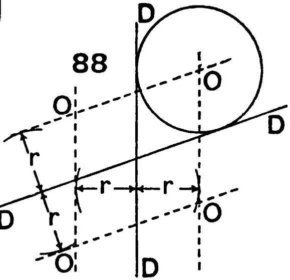
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Examples.—**1.** Draw a rectangle, sides 3.5″ and 2.5″. Also a parallelogram, sides 3.5″ and 2.5″, included angle 45°. Round off the corners of both figures by tangential arcs of 0.75″ radius. Show the junctions of the lines and arcs.

2. A curved line is made up of three circular arcs which join tangentially. The outer arcs each subtend 90°; their radii are 2″ and 3″; and their centres are 3.5″ apart. The middle arc is of 0.75″ radius. Draw the figure. Measure the angle subtended by the middle arc. Measure the total length of the curved line. *Ans.* 121° or 239°; 5.51″ or 7.06″.

3. Draw three circles in mutual external contact. Diameters 6, 4.5 and 3.5 cm. Show the points of contact.

4. Draw a triangle, sides 5″, 3.5″, and 3″. Draw a circle of ½″ radius on tracing-paper. Adjust the latter over the triangle so that the circle touches pairs of sides. Prick off the three centres. Then replace the corners of the triangle by tangential arcs of ½″ radius.

91. Theorem.—The angle in a semicircle is a right angle.—*Verification.* Draw any semicircle, diameter AA. Set out several angles ABA in the semicircle. Verify that in all cases the angle ABA is 90°.

Other experiments.—Insert two pins or needles as at C, C. Place the perpendicular edges of a set-square against the pins. Find the locus of the corner D as the square is moved into different positions.

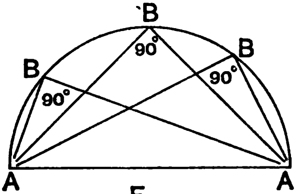
Again, mark two points on the drawing paper. Draw two perpendicular intersecting lines on tracing-paper. Superpose the latter over the former, one line moving over each point. Prick off the locus of the intersection of the lines.

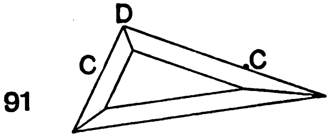
92. Theorem.—The angles in the same segment are equal.—Describe a circle, and draw any chord EE. Draw several angles EFE in the segment. Verify their equality by measurement or by using a tracing.

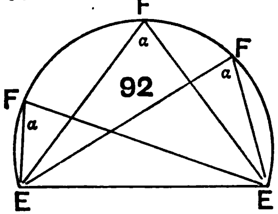
Make experiments similar to those described in Art. 91, but using the acute angles of the squares, and any intersecting lines on the tracing-paper, not forgetting an obtuse angle.

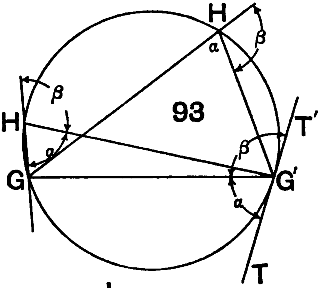
93. Theorem.—The angles made by a tangent with any chord from the point of contact are equal to the angles in the alternate segments.—First notice that the angles (marked β), between GH produced and HG′, being supplementary to the angles α in the segment, are all equal. Now suppose H to move up to and eventually coincide with G, then the extended chord GH becomes the tangent at G, and β becomes the angle between the tangent and chord. This is shown at G′ where the tangent TT′ is drawn. The angle GG′T, being supplementary to GG′T′ or β, is equal to α, the angle in the *alternate* segment. The angle in the lower segment would be β.

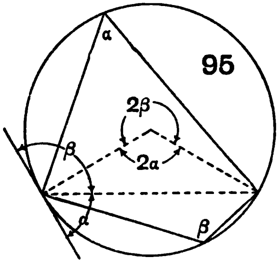
94. Theorem.—The angle in a segment is equal to half the angle at the centre. That is KLK is half KCK.—Draw Ck, Ck parallel to LK, LK. Now from the properties of parallel lines, and from the fact that the base angles of an isosceles triangle are equal, the student should easily make out that the four angles marked 1 are equal, and also those marked 2. Hence KCK = twice KLK.

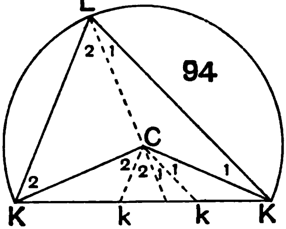












95. Theorem.—In any quadrilateral inscribed in a circle, opposite angles are supplementary, or their sum is equal to 180°.

Two ways of establishing this proposition are indicated in the figure. This figure sums up the properties stated in the previous four articles, and is worth attentive study.

Example.—Draw a circle of 2.25″ radius. In this circle inscribe a quadrilateral ABCD, having given :—Sides, AB = 2.87″; DC = 2.5″. Angle, BCD = 76.5°. Measure the angle BAD. Draw the tangent to the circle at A. Join AC and measure the angles which AC makes with the tangent. Also measure the angles ABC and ADC. (1901)

We now give several problems to illustrate the application of the foregoing principles.

96. PROBLEM.—**From a given point A on the circumference of a circle to cut off a segment which shall contain a given angle, say 50°.**

Draw the tangent at A (perpendicular to the radius). Draw the chord AA making 50° with the tangent. Then ABA is the required segment. See Art. 93.

97. PROBLEM.—**On a given line CC as chord, to construct a segment which shall contain a given angle, say 100°.**

From one end of the chord draw the line CD making 100° with the chord. The required circle must touch CD at C, and its centre must therefore lie somewhere in the perpendicular to CD from C. Draw this perpendicular.

But the circle passes through the two points C, C and its centre must therefore be some point in the perpendicular bisector of CC, which draw.

The intersection of the two loci of centres (the dotted lines) thus locates the centre E, and the arc of the required segment can be drawn.

98. PROBLEM.—**Having given two points P, P and a line LL, to find a point K in LL from which the points P, P shall subtend a given angle. Say PKP = 90°.**

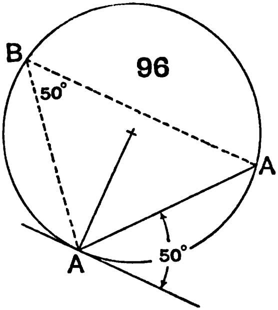
On PP as chord, construct a segment which contains the given angle. In this case a semicircle.

This segment intersects the line in the required point K. There are generally two solutions.

99. PROBLEM.—**To plot a circular arc which shall pass through three given points R, Q, R, the centre being supposed inaccessible.**

Draw the chords QR, QR. From one of the points R draw a series of lines. From the other point R, draw a second series to meet the first at a constant angle equal to the angle RQR. The intersections are points on the required arc.

This construction is most readily affected by tracing and



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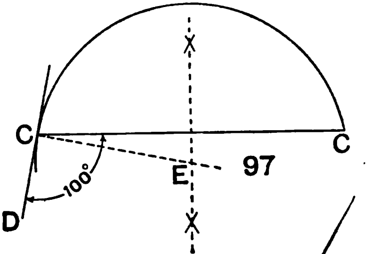

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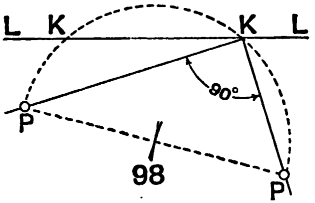

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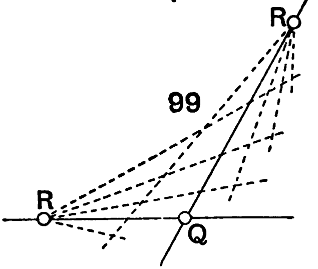

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extending the lines RQ, RQ, and then pricking off the point Q as the tracing-paper is moved into successive positions, with the pair of lines always passing over the pair of points R, R.

Observe that the tangents at R, R may be pricked off. Also that the arc may be extended beyond the points R. Let the student invent a method of drawing the tangent at Q, by using the tracing.

Examples 1.—Draw a triangle having given :—Base 2.6″, vertical angle 70°, altitude 1.6″. Measure its **perimeter,** that is, the sum of its three sides. *Ans.* 6.92″.

2. Copy (by means of tracing paper) the flat circular arc on page 95. Then draw accurately the tangent to the arc at the point R. Also, by the use of tracing paper, extend the arc for a length at least equal to itself in both directions.

3. Mark off three successive points A, B, C in a straight line, making AB = 1.5″, BC = 2.8″. Determine a point O such that the angles AOB, BOC shall be equal, and each 55°. Measure OA and OB. *Ans.* 1.79″, 3.34″.

4. Work Ex. 3 by means of tracing-paper. See the "three point problem," Art. 204.

100. Theorem.—If lines passing through a fixed point cut a circle, the product of the segments is constant.

Verification.—Draw any circle, centre C. Mark any inside point O. Join CO. Draw MOM perpendicular to CO. Observe that MM is bisected at O. Draw any chords AA′, BB′ etc., through O. Now confirm, by careful measurement and arithmetic that

$$OA \times OA' = OB \times OB' = OM \times OM \equiv OM^2 = \text{constant.}$$

Further, take any external point O. Draw a tangent from O and find the point of contact N. From O draw any lines ODD′, OEE′ etc., cutting the circle. Carefully verify that

$$OD \times OD' = OE \times OE' = ON^2 = \text{constant.}$$

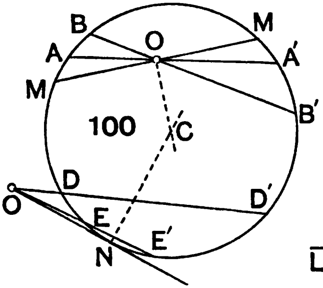
101. The arithmetical and geometrical means.—If we add two numbers together and divide by 2, the result is called the **arithmetical mean** of the numbers, or sometimes simply the **mean.** Thus the mean of a and b is $\frac{1}{2}(a+b)$.

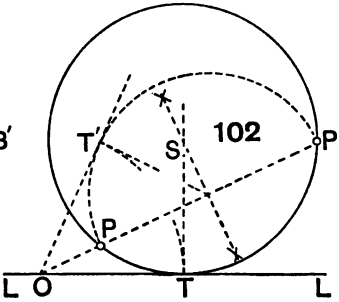
If we multiply the numbers together and extract the square root, the result is called the **geometrical mean,** or it is said to be a **mean proportional** between the numbers. Thus the geometrical mean of a and b is $\sqrt{ab}$.

If the numbers were 2 and 8, the arithmetical mean would be 5 and the geometrical mean 4.

Let a, b, c be three quantities in continued proportion (as *e.g.* 4, 6, 9) that is, let $a : b :: b : c$. Then, remembering that the product of the means is equal to the product of the extremes, we have $b^2 = ac$, or $b = \sqrt{ac}$. So that b is the geometrical mean between a and b. Hence also the appropriateness of the term mean proportional.

Observe in Art. 100 that OM is a mean proportional between OA and OA′, and between OB, OB′. Also ON is a mean proportional between OD, OD′ and between OE, OE′.





We give one problem showing the application of Art. 100.

102. PROBLEM.—**To draw a circle which shall pass through two points P, P and touch a line L L.**

We wish to find the centre S of the circle. First draw the perpendicular bisector of P P, this being a locus in which S must lie. If we knew the point of contact T of the circle with L L we could at once draw a second locus, and thus find S.

Let the line through P P cut L L in O. Suppose for a moment that the point of contact T were known; it is seen from the figure that O T is a mean proportional between O P and O P. Now draw *any* circular arc through P P, say a semicircle; draw the tangent from O, and find the point of contact T′. Then O T′ is also a mean proportional between O P and O P.

Hence make O T equal to O T′, and draw T S perpendicular to L L, locating S the centre of the required circle.

Examples 1.—Draw a triangle, base 2″, sides 1.8″ and 2.9″. Reduce this to an isosceles triangle having the same vertical angle. Measure the length of the equal sides. *Ans.* 2.28″.

Hint.—The product of the sides must remain the same. See Rule 6, p. 89.

103. Miscellaneous Examples on the Circle.

***1.** Section of an oval sewer, the outline being made up of circular arcs. Draw this figure to a scale of one inch to the foot. (1894)

***2.** Construct a figure made up of a trefoil of tangential arcs with the circumscribing circle, like the one shown, but to the dimensions given. Show all the construction lines clearly. *N.B.* No credit will be given for a mere copy of the figure. (1901)

***3.** Draw a figure like the one shown, but to the given dimensions. Show clearly all the construction lines. *N.B.* No credit will be given for a mere copy of the diagram. (1901)

***4.** Line made up of circular arcs and a straight line, which join tangentially. Construct this figure, having given :—

For AB, radius 1.3″, length of arc 1.8″.
For BC, radius 2.5″, angle subtended at centre 30°.
For CD, radius 2.2″, angle $\frac{1}{2}$ radian.
For DE, straight, length 1.6″.
For EF, length 1.3″, angle 1.5 radian.

Measure the total length from A to F, and the angle between the tangents at A and F. *Ans.* 7.1″, $5\frac{1}{2}$°. Also obtain these answers by calculation.

5. Draw a circle of 3″ diameter, and cut off a segment containing 80°. Measure the angle in the other segment. *Ans.* 100°.

6. Draw a square ABCD of $2\frac{3}{4}$″ side. Find a point P inside it so that the angle APB may be 90°, and the angle CPD 70°. (1898)

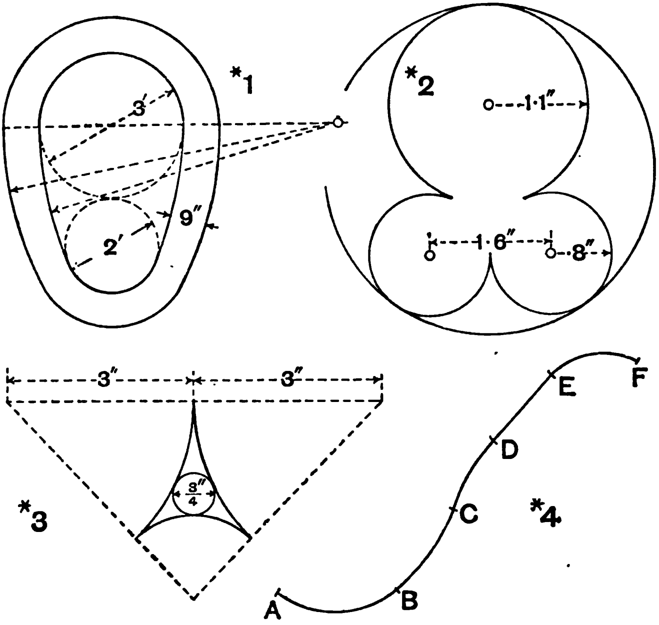
7. Draw a line AC, $3\frac{1}{2}$″ long. Through a point O in it, $1\frac{1}{2}$″ from A, draw a line perpendicular to AC. On this line let two points B and D lie, on opposite sides of AC. Draw the quadrilateral ABCD of which AC and BD are the diagonals, the angles ABC and ADC being 65° and 115°. (1899)

8. Draw a circle of $1\frac{1}{4}$″ radius, and a line distant $\frac{3}{4}$″ from its centre. Draw all the circles of $\frac{1}{2}$″ radius which touch both line and circle. *Note.* There are seven such circles.

9. Draw two circles of $\frac{3}{4}$″ and $\frac{1}{4}$″ radius having internal contact. Obtain the circles of $\frac{3}{8}$″ radius which have their centres on the larger circle, and which touch the smaller one (*a*) externally, (*b*) internally.

10. Obtain the radius of the flat arc on p. 95, (*a*) by finding the centre of the arc, (*b*) without finding the centre.

11. Draw an isosceles triangle, base 2.5″, sides 3″. Describe circles which shall pass through the vertex and the middle point of one side, and shall touch (*a*) the other side, (*b*) the base.



12. Find a mean proportional between two lines 3″ and 2″ long. Measure its length and verify the result by arithmetic.

CHAPTER X

THE CONSTRUCTION AND USES OF LOCI

104. Introduction.—Let the reader refer back to Art. 82, where the notion and meaning of a locus is explained and illustrated. All lines and curves, however originated, may be conceived as having been traced by a point, *i.e.* as being the locus of a point moving according to some law which guides or regulates the motion. The law may be very simple, as for the straight line and circle. Or, as in the case of a flowing curve described freehand by some artist, it may be so complex as to defy expression, and the curve be impossible of repetition except as a copy.

We are not concerned here with curves of undefinable form. But the elementary student of practical geometry should have some acquaintance with curves other than the circle. Curves are useful in various ways, and many of them are comparatively easy to draw, especially by the aid of tracing-paper.

As examples. We may wish to follow the movements of some part of a machine, or to trace the path of a moving body. An apparently difficult geometrical problem may often be solved quite easily by a person accustomed to the use of loci, by a method consisting essentially of a system of scientific guessing. Laws connecting two variable quantities which depend upon one another are often exhibited with great clearness by the plotting of curves.

Our treatment of curves can only be very cursory, considering the limited space available. We shall attempt nothing beyond that of giving a few typical examples suggestive of the construction and uses of curves.

In the actual plotting of any curve, except where a guiding template or mechanism is specially provided, the general method is to first determine a number of isolated points on it. These must be sufficiently near together to enable the continuous curve through them to be drawn freehand with the degree of precision that may be desired under the particular circumstances of the case. For the shape of the curve between the points we depend on the law of **fair curvature**, of which the eye is the competent judge, trained as it is amongst the beautiful curves of nature surrounding us on all sides.

105. Elementary examples of loci.—In order to make sure of our ground, we will repeat some of the examples of loci previously given in the last chapter, and add a few others for exercise.

1. What is the locus of a point which moves so that :—

- **(a)** It is always at a constant distance a fixed point O? *Ans.* A circle, radius a, centre O.
- **(b)** It is always at a constant distance b from a fixed line LL? *Ans.* Either of the two lines parallel to LL at distance b.
- **(c)** It is always at a distance c from the circumference of a circle, radius r? *Ans.* Either of the concentric circles, radii $r \pm c$.
- **(d)** Its distances from two fixed points, P, P are always equal to one another? *Ans.* The perpendicular bisector of PP.
- **(e)** Its distances from two intersecting lines are always equal to one another? *Ans.* Either of the bisectors of the angles between the lines. Note that these two loci are perpendicular to one another.

2. Draw the locus of a point which is always equidistant from a line LL, and circle, centre O, radius 1.2″. Distance of line LL from centre O of circle 1.6″.

Note.—As an alternative to the use of compasses, the student may draw a number of circles, increasing in size, on tracing paper. Then superpose these in succession so that they touch the line and circle, prick off the centres, and draw the required locus through the points so determined.

3. Draw any three circles. Then determine a circle to touch all three.

106. PROBLEM.—**A line PQ moves so that the point P describes a circle, Q always lying in an extended diameter of the circle. Draw the locus of any point K in PQ.**

Many students will here recognise the geometrical equivalent of the motion of the connecting rod in a steam engine.

The best method of proceeding is to mark the points P, Q, K on tracing-paper. Superpose this tracing on the circle with a diameter produced, in a number of positions all complying with the given law of the motion. In each case prick off K. Through these points draw a fair curve for the required locus.

107. PROBLEM.—**A line ab moves so that a and b always lie in two perpendicular lines BB, AA. Plot the locus of any point c in ab or ab produced.**

As before, mark the points a, b, c on tracing-paper. Superpose this tracing on the lines AA, BB so as to fulfil the conditions, and prick off the point c.

Repeat this operation as often as may be necessary, and through the points so determined draw the required locus.

The ellipse.—The moving line is known as an **elliptic trammel.** The locus is an **ellipse.** AA is the **major axis** of the ellipse, and BB is its **minor axis.** C is its **centre.** Observe that ca = CA, and cb = CB.

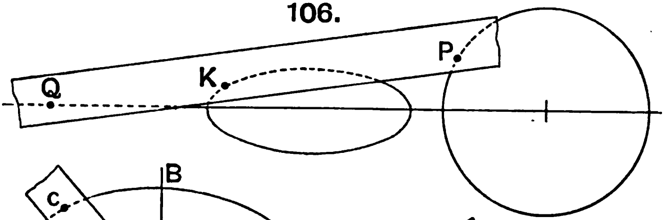
With centres B, B, radius CA, describe arcs intersecting in F, F. Each of these points is a **focus** of the ellipse. Suppose H to be *any* point on the curve (not shown). Join H to the foci F, F. Let the student verify the important property that FH + HF = AA. Or *the sum of the focal distances is constant and equal to the major axis.*

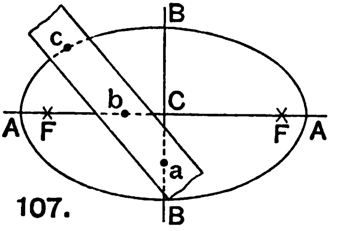
The reader who cares to think about this curve should be able, by use of the trammel, to construct an ellipse having given (1) *the major and minor axes, or* (2) *the foci and either axis.*

108. PROBLEM.—**A circle rolls without slipping on a straight line LL. Draw the locus of a point R on the circumference.**

Draw the circle on tracing-paper. To ensure rolling use the pricker at the point of contact. Proceed by short steps, shifting the pricker a little each time. At each step prick off R. The locus is the well-known **cycloid.**

106.





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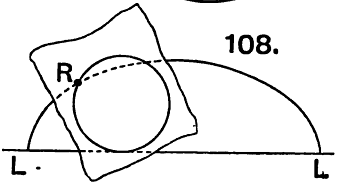


Any rolling curve may be very readily and accurately plotted by this use of tracing-paper and a pricker.

Examples.—**1.** Work problem 106, taking the diameter of the circle = PK = KQ = 2″.

2. Work problem 107, taking ca = 1.5″, cb = 1″. Find the foci of the ellipse. If H be any point on the curve, verify that FH + HF = AA.

3. Plot the locus of a point the sum of whose distances from two points, 2.5″ apart, is 3.5″.

4. Plot the locus of a point the difference of whose distances from two points 1.8″ apart is 1.5″.

5. Draw any two intersecting lines. And on tracing-paper mark any three points. Let two of the points move, one on each line. Prick off and plot the locus of the third point.

Note.—This curve is an ellipse.

6. Work problem 108, taking the diameter of the rolling circle 1″.

7. A line rolls on a fixed circle $\frac{3}{4}''$ diameter. Plot the locus of any point in the line.

Note.—This curve is called the **involute of the circle.**

8. A circle $\frac{3}{4}''$ diameter rolls on a circle 2″ diameter. Determine the path traced by a point on the circumference of the rolling circle, when the contact is (*a*) external, (*b*) internal.

Note.—These two curves are called the **epicycloid** and **hypocycloid** respectively.

109. PROBLEM.—**A circle and internal point O are given. It is required to draw through O a chord AOB, such that the segments OA, OB shall be in a given ratio, say 2 : 3. That is, OB = 1.5 OA.**

The discovery of a solution to this problem on Euclidean lines would probably take a considerable time. A solution by means of a locus should be quite obvious.

First choose what seems a promising point A_1; draw and produce the line A_1O, and mark off $OB_1 = 1.5\ OA_1$. This is our first guess. If B_1 accidentally falls on the circle, our work is done.

We are now in a position to give a better guess. A_2 appears more promising than A_1. The result B_2 justifies our expectation, but is still only a guess.

Our third, and what should be the last trial, is A_3, yielding B_3.

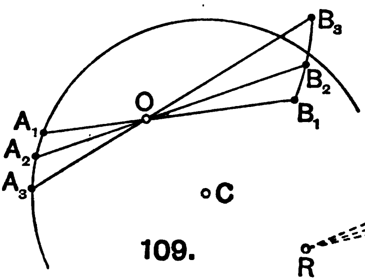
Now join the three points B_1, B_2, B_3 by a fair curve. That is, find the locus of B. The intersection of this locus with the circle gives us the point B of which we are in search.

The drawing of the locus transforms our method of trial into a scientific process.

The student who remembers how figures are enlarged by radial projection, may now perhaps recognise that $B_1\ B_2\ B_3$ is a magnified and inverted image of $A_1\ A_2\ A_3$, O being the pole; thus the locus is a circular arc. To locate its centre, find the image of C, by joining and producing CO to C′, making OC′ = 1.5 OC.

110. PROBLEM.—**R and S are two points, and MM is the section of a convex spherical mirror, centre O, by the plane through R, O, S. Find the path of a ray of light, which, proceeding from R and being reflected by the mirror, would pass through S.**

A law of optics tells us that the angle of reflection is equal to the angle of incidence. In other words, the two branches of the ray must make equal angles with the circle



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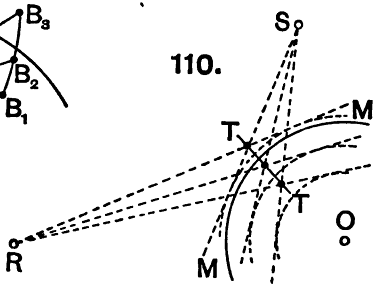

110.

at the point of contact, that is, equal angles with the tangent or radius at this point.

The search for a method which shall involve the drawing of only straight lines and circles would prove a long one, but to the attack of a locus the problem yields quite readily. There are several loci which would serve; we give one. If the student wishes for the key to the solution, let him observe that as the two lines of the ray make equal angles with the radius, they would, if produced inwards, both touch the same circle, centre O.

Draw any circle, centre O. Draw tangents from R and S, intersecting in T. We wish T to fall on the mirror. So repeat this process for other circles; plot the locus of T; and find the point where this locus intersects the mirror M M.

Examples.—1. Draw two lines Oa, Ob including 40°. Mark a point P outside this angle and adjacent to Oa, distant 3.5″ from O and 1.5″ from Oa. Then through P draw a straight line to cut the lines Oa, Ob in A and B, so that the ratio PA : PB shall be 2 : 3. A locus may be used if desired.

2. Draw a circle of 1.5″ radius. Mark a point O distant 2.5″ from its centre. Through O draw a line to cut the circle in A and B, so that OA shall be half OB.

3. Draw a triangle ABC, making the lengths of the sides AB = 12 cm., BC = 9 cm., CA = 6 cm. Determine a point O distant 6.5 cm. from C such that the angles AOC and BOC shall be equal.

111. The equation to a curve.—Let O be the centre of a circle, P a point on the circumference. Now wherever P may be on the circle we have always $OP = \text{radius} = \text{constant}$. Say $r = a$.

This equation may be regarded as a shorthand way of stating the property that *a circle is the locus of a point which always remains at a constant distance* a *from a fixed point* O. It specifies the property in the terse form of an equation. It is an **equation to the circle.**

Continuing, through O draw two perpendicular lines of reference OX, OY. On these lines drop perpendiculars PN, PM. As in Art. 56 we may call the lengths of these perpendiculars x and y. They are the rectangular co-ordinates defining the position of P.

Now in the right-angled triangle OMP we have:—$OM^2 + MP^2 = OP^2$. That is, $NP^2 + MP^2 = OP^2$; or $x^2 + y^2 = a^2$.

This gives us another **equation to the circle** in terms of rectangular co-ordinates. The equation is true wherever P may be on the circumference. Interpreted, the equation tells us that *a circle is the locus of a point which moves so that the sum of the squares of its distances from two perpendicular lines is constant.*

It can be shown that all curves of determinate form have equations which define their shape.

112. Plotting a curve from its equation.—Suppose a point to trace some definite curve in a plane. In order to be able to locate its position wherever it may happen to be situated, let us take any two perpendicular lines OX, OY, and call the distances from these lines y and x as before. Thus if we know x and y, we can plot the point; or if the point be known on the diagram, we can measure its x and y.

Now the point is not permitted to wander anywhere as if it were a horse grazing. It must keep to the path like the unfortunate pedestrian.

This restriction of its freedom implies some law or relation connecting x and y. In our case let us suppose

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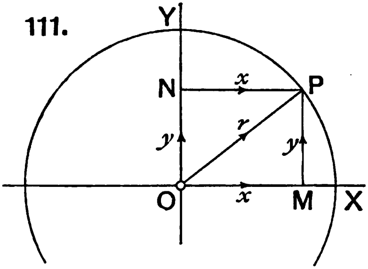

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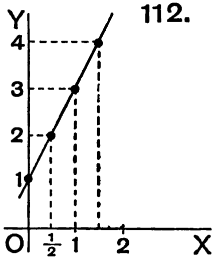


the law can be expressed as a simple equation between the co-ordinates, such as $y = ax + b$. Say $y = 2x + 1$.

To find points on the locus we may assign any values to x and calculate the corresponding values of y.

Thus when $x = 0, y = 1$; when $x = \frac{1}{2}, y = 2$; when $x = 1$, $y = 3$; when $x = 1\frac{1}{2}, y = 4$, and so on. We see that y increases twice as fast as x and has a start of 1.

These four points are shown plotted on the diagram. It is evident that the locus is a straight line.

If the axes were two hedges of the field in which the cattle were grazing, the path $y = 2x + 1$ to which the public are restricted would enter the field through the fence OY at distance unity from the corner. It would go obliquely across the field in a direction making $\tan^{-1} 2$ or 63.5° with the hedge OX.

Examples.—Plot the following curves from their equations :—

1. $y = 0.5x + 1$; $x = 1.5y$. Straight lines.
2. $y = 0.2x^2 + 1$. A parabola.
3. $xy = 2$. A hyperbola.
4. $y = 2 \sin x$. A sine curve.

Note on the use of squared paper.—In the Practical Mathematics class, in the laboratory, and elsewhere, the student should have become accustomed to the plotting of curves on squared paper, the horizontal and vertical divisions of the paper being designed to facilitate the setting off of rectangular co-ordinates. It might be found advantageous to introduce some amount of this kind of work along with drawing.

CHAPTER XI

VECTORS

113. Directed quantities.—Whenever we go from one place to another; when we play a game like cricket or tennis; when any object falls to the ground, or even remains at rest; as a top spins, or the earth turns on its axis; when a magnet is placed in a field of force. In these and in a hundred other cases, surrounding us on all sides, we have, intimately associated with one another, *magnitude* and *direction in space*. Quantities possessed of these two attributes may be called **directed quantities.**

Thus between two places there is not only the distance apart, which is a magnitude, but there is also the direction from the one to the other which may be east or west, north or south, or some other direction. In playing cricket, the ball, as it flies through the air, is endowed with speed and direction of motion; the batsman thinks how hard he shall hit the ball, and at the same time of where he shall place it. A body falling freely moves with a *downward speed*, under the action of gravity; and when at rest there is still the ever-present force of gravitation, so many pounds, directed towards the centre of the earth. A spinning top, like the earth, is turning at some definite rate, and about some axis which has direction in space. And the magnet is under a force action of determinable amount, and which tends to make it take up a definite direction in the magnetic field.

Some quantities are essentially *non-directive*, such as area, volume, bulk, mass, energy, temperature, etc. In these we have only magnitude to consider. In other cases, although we may be dealing with directed quantities, yet the conditions may be such as to require only one aspect to be taken into account. For instance we might have a problem involving the speed of a railway train, the direction of motion being ignored; or another problem requiring us to know the direction of the sun's rays, but taking no notice of the sun's distance, or the speed of light.

In most of the problems which have been considered in the preceding chapters we have been concerned mainly with the magnitudes of lines, angles, and figures, irrespective of any properties of direction which these might possess. In the present section we are to pay special attention to the association of magnitude and direction in one and the same quantity, and it is a distinctive feature of the problems that both factors shall come into the account.

114. How directed quantities may be represented.—To take a simple case, suppose we wished to specify the position of a point in a plane relatively to another point in the plane. For instance, how might a resident in Liverpool describe the position of Manchester? It would be sufficient to state that Manchester was 30 miles distant in a direction 10° to the north of east. Or, as it might be written, 30 miles E. by 10° N., or 30 miles E. 10° N. This is a statement of a length and an angle and is the **numerical specification** of the relative position in question.

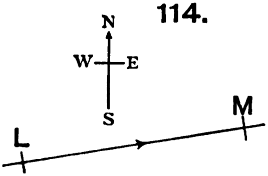
In the **graphical representation**, we should first indicate the cardinal points of the compass, or mark say which was East. We should then draw a line in the proper direction, and cut off a segment 30 miles long to scale, specifying the latter. And finally we should place an arrowhead on the segment, pointing towards Manchester, to indicate that this, and not the opposite direction, was the one with which we were concerned. Instead of the arrow, we might write letters as L, M at the ends of the segment, and state that the direction was from L to M, or was the direction LM. (ML would denote the opposite direction from M to L.)

The above methods of representation are typical, and the reader will see as we proceed that all the directed quantities with which we shall deal can be defined by the statement of a magnitude and an angle, or by drawing the directed segment of a line to scale.

115. To define the position of a point in a plane.—In order to do this first mark any fixed *point of reference* O in the plane. Then from O draw a fixed *line of reference* in any direction, say OX eastward. Now, in defining the position of any point A in the plane, imagine a line, starting from the position OX to rotate about O, say in the anti-clockwise direction, until it passes through A, thus describing an angle which we will call α. Then suppose a point beginning at O to travel along the line until A is reached, requiring a distance A to be moved through. The position of A, a directed quantity, is thus fixed by the statement of the length A and the angle α. It is sometimes convenient to write the angle as a suffix to the length, in the form, position of $\mathrm{A} = A_{\alpha}$. Two other points B and C are shown in the figure. C's position is 41_{208° feet from O, as can be verified by measuring the diagram to the specified scale. For the position of B see Ex. 2.

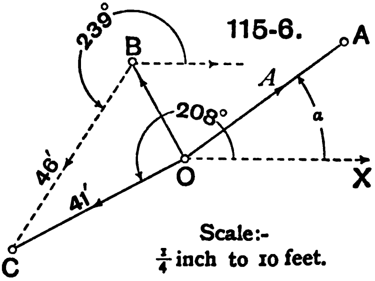
116. Change of position or displacement.—If a point were to change its position in the plane, this displacement would be a directed quantity, and could be represented in the same way that position itself is represented. Thus a displacement from O to A would be indicated by the directed line OA measured to scale, or by the statement—displacement $= A_{\alpha}$. A change of position from B to C could similarly be represented by the directed line BC, or by the numerical specification—displacement $= 46_{239^\circ}$ feet, the direction of the displacement being defined by the anti-clockwise angle of 239° which its line makes with the eastward direction, this being understood as the direction of reference. This should be quite clear from the figure.

114.



Linear Scale:-
1 cm. to 10 miles.

115-6.



Scale:-
$\frac{1}{4}$ inch to 10 feet.

116a. Linear velocity.—If the displacement were continuous and constant, and if the line representing it referred to the change of position that occurred in unit time, then this would indicate the **velocity** of the point. For example, if the unit of time were 1 second, B C would represent a displacement of 46 feet for each second in the direction from B to C, that is, a velocity of $46_{239°}$ feet per second.

Examples 1.—Mortlake is 2 miles west of Putney, and Wimbledon is 3 miles south of Putney. Plot the relative positions of these places to a scale of 1 inch to the mile, and measure the distance and direction of Wimbledon from Mortlake. *Ans.* 3.6 miles S. 33.7° E.; or, as in Art. 115, measuring angles anti-clockwise from the east, the position of Wimbledon relatively to Mortlake $= 3.6_{303.7°}$ miles.

2. In the figure above, measure the position of B referred to O X, and represented by O B. *Ans.* $23'_{120°}$.

3. If O A in the figure above represents a displacement, measure its elements A and a. *Ans.* Displacement $= 41'_{38°}$.

4. Represent the following velocities by directed lines :—

(*a*) 43 feet per second in direction 50° north of east. Scale 1 inch to 10 feet per second.

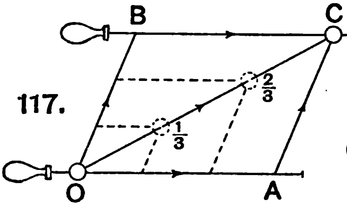
(*b*) A velocity of $50_{270°}$ miles per hour. Scale, 1 cm. to 10 miles per hour. Take O X and measure angles as above.

5. A train goes from Manchester to Liverpool, a distance of 30 miles in 45 minutes; find its average *speed.* *Ans.* 40 miles per hour. What is its velocity? *Ans.* $40_{190°}$ miles per hour. Represent this velocity graphically, and state the scale employed.

117. The parallelogram of displacements.—Let OA be a straight wire on which a bead can slide, and let O be the initial position of the bead. Now move the bead along the wire from O to A, and at the same time move the wire parallel to itself into the position BC. It is quite evident that at the end of the operation the bead will be at C. It will have moved from O to C. Observe that OBCA is a parallelogram; and that opposite sides of parallelograms are equal in length.

In the above process the bead has had imparted to it two **component** displacements; one, represented in magnitude and direction by either of the directed lines OA and BC; the other similarly represented by either OB or AC. The **resultant** displacement is represented by the diagonal OC, directed from O to C. This construction is called the **parallelogram of displacements.** Note carefully that OA, OB and OC, the line representatives of the component and resultant displacements are all **directed outwards** from O; also that the equivalent representatives BC, AC and OC are **all directed inwards** towards C.

We must distinguish between the *resultant* displacement and the actual *path* of the point. When the component movements of bead and wire take place simultaneously with a constant velocity ratio, the path of the bead is along the diagonal OC. Thus the points marked $\frac{1}{3}$ and $\frac{2}{3}$ show intermediate positions of the bead when these fractions of the two displacements have been effected. But the displacements might take place in succession, and in either order. If the bead were first moved along the wire, and then the wire shifted, the path of the bead would be OA, AC. If the wire were moved first and the bead next, the path would be OB, BC. Other paths might be traced; in fact the bead could be made to follow any path whatsoever, by suitably timing and varying the rates of movement of bead and wire. But in all cases, when the component changes of position represented by OA and OB have been completely effected, the bead will be at C, and the diagonal OC will always represent the **resultant** change of position, quite independently of the actual path of the point. In problems on vectors we are seldom concerned with the actual paths, but only with the final change as represented by the diagonal. It is as if a person goes a journey to a place, and we are only concerned as to his getting there, paying no attention to his particular route.



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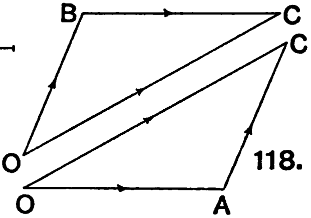

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118. The triangle of displacements.—Instead of the complete parallelogram, we could, knowing the sides, obtain the diagonal OC by drawing one or other of the triangles OAC or OBC, as shown detached to the right. Either of these triangles is called the **triangle of displacements.**

Note carefully that the given components OA, AC, or OB, BC follow one another in *tandem* or *follow-my-leader* fashion, or **circuitally**; and that the resultant OC is directed non-circuitally, or against the circuit of the components.

It may be of assistance if we think of the two triangles as the result of the component displacements taking place in succession. In one the order is OA, AC. In the other it is reversed, being OB, BC.

Examples 1.—Find the resultant of the following component displacements:—

(*a*) 3_{0° and 2_{60°. *Ans.* $4.36_{23.5^\circ}$.
(*b*) 1.5_{0° and 2_{90°. *Ans.* $2.5_{53.1^\circ}$.
(*c*) 47_{35° and 59_{116°. *Ans.* $80.7_{81.4^\circ}$.

2. Find what displacement compounded with one of 3_{20° will result in a displacement of 4_{50°. *Ans.* 2.05_{97°.

3. A person journeys to a place 2.5 miles to the east. He does this in two stages, both straight, the first 2 miles long, and the second 1.5 miles long. Find the directions of the two parts of the journey. *Ans.* -37° and 53°, or 37° and -53°.

4. A railway train 312 feet long is observed to occupy 12.4 seconds in passing a certain point; calculate the speed of the train. *Ans.* 25.2 feet per second. If its direction of motion is N. *b.* W., what is the velocity of the train? *Ans.* $25.2_{101\frac{1}{4}^\circ}$ feet per second. Represent this velocity graphically in any manner which shall be free from ambiguity.

119. Meaning of the term vector.—Instead of receiving component displacements, as explained in the last article, a point might be the recipient of two velocities, or be subject to the action of two forces. It will be seen presently that in these cases the velocity, or the single force equivalent to the two forces, can be found by drawing a parallelogram of velocities, or a parallelogram of forces, exactly like the parallelogram of displacements. The word **vector** is a general term used for all directed quantities like displacements, velocities, forces, etc., which are **compounded by or subject to the law of the parallelogram.**

It is sometimes affirmed that all quantities possessed of magnitude and direction may be combined by the parallelogram rule, and that the terms "directed quantity" and "vector" are therefore synonymous. This statement, however, is of too sweeping a character. A poplar tree has length and a very pronounced direction, and in a sense may be regarded as a directed quantity. Yet the parallelogram rule would be meaningless in this connection; the tree is not a vector.

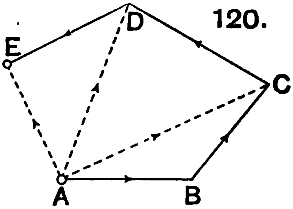
120. The parallelogram, triangle, and polygon of vectors.—The law of the parallelogram is of such fundamental importance, that, although we have already given it in reference to displacements, and it is the same for all vectors, yet we shall restate it.

The vector parallelogram.—*The resultant (or sum) of two vectors is given by the diagonal of a parallelogram having the two vector lines as adjacent sides, such that the two sides and the resultant are directed all away from, or all towards their common end point.*

The vector triangle.—*To obtain the resultant (or sum) of two given vectors, draw the vector lines end to end circuitally, and obtain the non-circuital closing side of the vector triangle.*

The vector polygon.—*To obtain the resultant (or sum) of any number of vectors (of like kind) draw the vector lines end to end circuitally, and obtain the non-circuital closing side of the vector polygon.*

The **vector polygon** may be regarded as an extension



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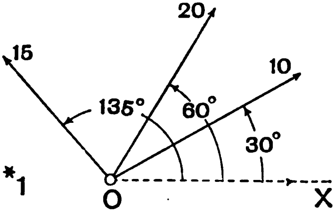

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of the vector triangle. Thus in the figure let AB, BC, CD, DE be the vector lines arranged end to end, all circuitally; then AE is the non-circuital closing side, and AE is the resultant of AB, BC, CD, DE.

For, by the rule of the triangle, AC is the resultant of AB and BC; and again, by the same rule, AD is equivalent to AC and CD, that is, to AB, BC, and CD; and finally, AE is the resultant of AD and DE, that is, of AB, BC, CD, and DE.

Or the truth of the proposition is at once evident if the vectors be regarded as displacements. The proposition is equivalent to saying that if, starting from A, a journey be taken round the sides of the polygon to E, the resultant change of position is AE.

Examples.—***1.** Find the resultant or sum of the vectors 10_{30°, 20_{60°, and 15_{135°. See the figure. *Ans.* $32.75_{103.8^\circ}$.

2. A person walks 29 steps to the E., 51 steps to the S.S.E., and 13 steps to the S.W. Find how many steps respectively W. and N. will bring him back to his original position. *Ans.* 39.3, 56.7.

3. A ship at sea sails 8.7 miles through the water, apparently to the E., but an ocean current carries the vessel 3.4 miles to the S.-W. Find the resultant movement of the ship, *i.e.* its change of position as regards the ocean bed. *Ans.* 5.88 miles E. 20.9° S.

4. A boy on a raft walks 9 feet from a point A to a point B, and during this time the raft moves (without rotation) a distance of 14 feet in a direction making 115° with AB. Find the resultant movement of the boy. *Ans.* 13.1 feet, making 76.4° with AB.

5. In Ex. 4, suppose the movement of boy and raft to occupy 2 seconds. Calculate the speed of the raft and the speeds of the boy as regards the raft and the bed of the stream. *Ans.* 7, 4.5 and 6.55 feet per second.

121. The closed vector polygon.—We have seen that if a series of vectors be arranged end to end circuitally, as AB, BC, CD, DE, beginning at the point A and terminating at E, then the resultant is given by the non-circuital vector AE, closing the polygon from the initial point A to the end point E. Now if E happen to coincide with A, the resultant AE is evidently zero, and the sides of the polygon would all be directed circuitally.

Or if the vector EA, equal and opposite to the resultant AE, were added to the system, the point A would now be both starting and end point, and the resultant effect would again be nil. The circuital side EA, closing the polygon from the terminal to the initial point is called the *equilibrant* or the *balancing vector.* It nullifies the effect of the others, and reduces the system to zero or to equilibrium.

We will now state the above propositions formally.

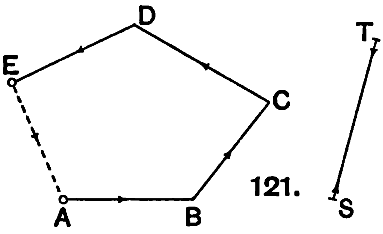
Equilibrant.—*The equilibrant of a system of vectors is given by the circuital closing side of the vector polygon.*

Theorem.—*A system of vectors represented by the circuital sides of any closed polygon is in equilibrium.*

It will be sufficient at present if the student understand these principles in reference to displacement vectors. The circuital closing side EA represents simply the displacement which would lead back to the starting-point. And displacements represented by any closed circuit bring the traveller home again, there being no final or resultant change of position. The components neutralise one another, leaving affairs unchanged.

As a simple case of velocity vectors, if a swimmer were to go against a stream at the exact rate of the stream itself, he would remain stationary as regards the bank. The vector lines representing the two velocities of stream and swimmer would in this case coincide, but be oppositely directed, as ST, TS. See the figure.

122. Rectangular or perpendicular components of a vector.—To take a simple illustration, suppose a person, starting from a place P reaches a place Q by walking a distance PM of 3 miles to the east, and then a distance MQ of 4 miles to the north. . He would evidently reach the same plane by first walking 4 miles PN to the north,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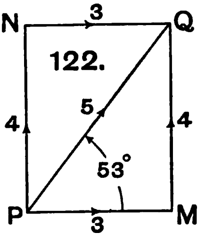
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and then 3 miles NQ to the east. And the student will readily make out that Q is 5 miles away from P in the direction E. 53° N., that is, 53° north of east.

PM and PN, or NQ and MQ are the easterly and northerly components of PQ. The vector might be replaced by or resolved into these components. If PQ were a force of 5 lbs., the same effect would be produced by the component forces PM and PN of 3 lbs. and 4 lbs., the force of 5 lbs. being removed.

The vector PQ could have been resolved into components in other perpendicular directions, say N.E. and N.W. Let the student verify that in this case the components would have been 0.695 and 4.94 miles.

Also any vector can be resolved into two components not at right angles, and indeed into any number of components, and in endless variety, just as the route between two places can be infinitely varied. For example, in the figure on p. 123, the diagonal OC can be resolved into the sides OA and OB. And in the figure on p. 125, AE could be resolved into and replaced by AB, BC, CD, and DE.

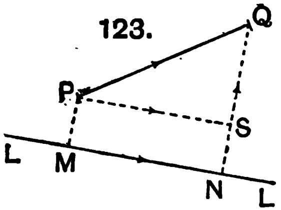
But there is a unique way of resolving a vector, of such importance and frequent application that we shall give it as a special problem in the next article. See Prob. 123.

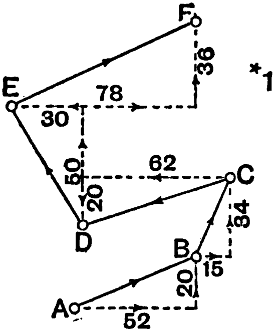
Examples.—1. A football is kicked a distance of 52 yards in a direction making 35° with the side lines of the field. Find the component displacements of the ball forwards and sidewise respectively. *Ans.* 42.6 yards, 29.6 yards.

2. A person journeys 1 mile eastwards and $\frac{3}{4}$ mile to the north-east; find his resultant change of position. *Ans.* 1.62 miles E. 19.1° N.

Find also the northerly and easterly components of the resultant. *Ans.* 0.53 and 1.53 miles.

123.





Plotting from Latitude and Departure.

	N	S	E	W
AB	20		52	
BC	34		15	
CD		20		62
DE	50			30
EF	36		78	
Totals	140	20	145	92
AF	120		53	

diagram from these new numbers, thus avoiding transmitted errors which might accumulate.

2. A body receives component displacements of 2, 5, and 7 units in the directions of the circuital sides of an equilateral triangle. Find and measure the resultant displacement. *Ans.* 4.3 units, making 83.4° and 36.6° with the 5 and 7 displacements.

3. Reckon angles as in Art. 115, anti-clockwise from the east, that is northwards of east. A river flows with a velocity of 24_{0° yards per minute. A man on a barge is situated 9_{125° yards from the centre of the latter, and in one minute, by means of a pole, he propels the barge through the water a distance of 15_{70° yards (*i.e.* relatively to the stream), during which time the barge turns about its centre through an angle of -60° (*i.e.* clockwise). Find the resultant displacements of barge (centre) and man which have occurred during this interval of time. Adopt a scale of $\frac{1}{8}$ inch to 10 yards. *Ans.* $32.0_{24.2^\circ}$, $40.6_{15.3^\circ}$ yards.

CHAPTER XII

VECTOR ADDITION AND SUBTRACTION

125. Recapitulation of Chapter XI.—We have seen that many quantities around us are possessed of magnitude and direction inherently associated, and are subject to a rule of equivalence, viz. the law of the parallelogram, such directed quantities being known generally as vectors.

A vector in a plane may appropriately be represented on paper by a directed segment of a line drawn to a suitable scale, or defined numerically by stating its two elements, magnitude, and direction, in the form $\mathbf{A} = A_a$ where $\mathbf{A}$ denotes the vector itself, A being its magnitude or **scalar** part, and a its direction referred to some fixed direction.

Any fixed direction may be taken as that of reference, and angles may be estimated either way round. But in what follows it is to be understood, unless otherwise stated, that angles defining directions are always to be measured **anti-clockwise from the east**, that is northwards of east.

Thus the direction due north would be the direction 90°. And due south would be 270° or $-90°$, because in order to bring the directed line of reference so as to point south would require an anti-clockwise or positive rotation of 270°, or a backward or negative rotation of 90°. The direction west would be either $+180°$ or $-180°$. The direction 315° would mean S.E., and $-45°$ would denote the same direction.

A difference in direction of 360° corresponds to the same direction; thus $A_{\alpha+180°} = A_{\alpha-180°}$. Two vectors whose directions differ by 180° are parallel, but they act opposite ways. For example, if A_α were the resultant of any system of vectors, then $A_{\alpha+180°}$ or $A_{\alpha-180°}$ would be the equilibrant of the system.

Thus the two vectors $4_{30°}$ and $4_{210°}$ if they acted together, would cancel one another; one is the negative of the other. We might appropriately write $4_{30°} = -4_{210°}$, or $A_\alpha = -A_{\alpha\pm180°}$ or 4 south = − 4 north. That is, the negative of a vector is a vector of the same magnitude, but reversed in **sense.** This would be indicated on a drawing by simply reversing the direction arrow.

The student should notice that the symbol = when used in reference to vectors implies not only equality in magnitude but also the same direction. A walk of 2 miles to the east is not the same as a walk of 2 miles to the north, or 2 miles to the west. We could not write $2_{0°} = 2_{90°}$ nor $2_{0°} = 2_{180°}$, although the magnitudes are equal. And the statement or equation $A_\alpha = B_\beta$ implies that α and β are equal as well as A and B. And the expression $\mathbf{A}=\mathbf{B}$ implies **vector equality** between **A** and **B**, *i.e.* equal magnitudes having the same direction.

Vector addition. We have seen that to obtain the resultant of any system of vectors, we must place them **end to end circuitally.** It will be very convenient to call this process that of **adding** the vectors. The **sum** would be another name for the resultant, and would mean the non-circuital closing side of the vector polygon. Thus in the figures on p. 127 we could write

$$\mathrm{AB}+\mathrm{BC}+\mathrm{CD}+\mathrm{DE}+\mathrm{EA}=0. \quad \mathrm{ST}+\mathrm{TS}=0. \quad 3_{0°}+4_{90°}=5_{53°}.$$

Examples.—1. Represent to suitable scales the following vectors, in each case stating the scale:—

10 miles S.S.W.; $37_{45°}$; $2_{-60°}$; $-2_{60°}$; $-2_{-60°}$.

2. Verify the following vector equations:—

2 N. + 2 E. = 2.83 N.E. 4 E. + 3 E. = 7 E. 4 N. + 3 S. = 1 N.

10 E. 50° N. = 10 cos 50° E. + 10 sin 50° N. = 6.43 E. + 7.66 N.

$2_{0°}+3_{90°}=3.6_{56°}$. $2_{20°}+3_{75°}+4_{210°}=1.68_{101°}=0.61_{180°}+1.58_{90°}$.

126. The sum of a series of vectors is the same in whatever order the components are added.—The student knows this to be true for mere numbers; that is 3 added to 4 gives the same answer as 4 added to 3. And from the figure on p. 127 he can see that PM+MQ=PN+NQ = PQ, or 3 E. + 4 N. = 4 N. + 3 E. = 5 E. 53° N. Let the reader satisfy himself that the rule is always true.

As an illustration, let the component vectors be **A**, **B**, **C** or $3_{15°}$, $8_{40°}$, $5_{160°}$, as shown at (*a*), the directions being plotted and the magnitudes figured.

At (*b*), to a scale of $\frac{1}{8}''$ to 1 unit, we show these vectors added in the sequence **A** + **B** + **C**, and also in the sequence **A** + **C** + **B**. The result is seen to be the same in both cases, viz. **D** or $8.8_{61°}$. Thus

$$3_{15°} + 8_{40°} + 5_{160°} = 3_{15°} + 5_{160°} + 8_{40°} = 8.8_{61°}.$$

Let the reader verify this result by actual drawing to a scale of 1 cm. to 1 unit, plotting the angles carefully, and taking sequences different from those shown.

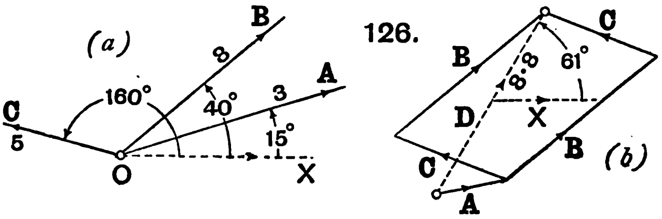
127. Vector subtraction.—The vector − **A** means **A** reversed in sense, or turned through 180°. The vector difference **B** − **A** means **B** + (− **A**), or **A** reversed in sense and added to **B**, or what comes to the same thing, **A** added non-circuitally to **B**. Otherwise $B_\beta - A_\alpha = B_\beta + A_{\alpha+180°}$.

As an illustration we have shown at (*c*), to a scale of $\frac{3}{16}''$ to 1 unit, the value of **A** − **B** − **C** of (*a*), that is, of $3_{15°} - 8_{40°} - 5_{160°}$. The result measures $6.4_{-76°}$ or $6.4_{284°}$.

128. The bracket law of algebra, represented by the equation A − (B + C) = A − B − C, is also true for vectors.

In Fig. (*d*) we have found **B** + **C**, viz. $8_{40°} + 5_{160°}$, and added it, with reversed sense, to **A**. The result is the same as in (*c*), viz. $6.4_{-76°}$.

129. Vector equations.—It will be found on examination that in a vector equation a term may be transferred from one side to the other, accompanied by a change of sign, just as in algebra, *e.g.* $3_{0°} + 4_{90°} = 5_{53°}$ could be changed to $3_{0°} = 5_{53°} - 4_{90°}$.



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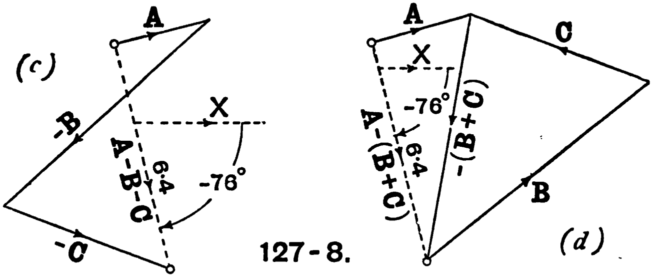


127-8.

Summing up our results, it appears that, with the extended meanings given to the symbols +, −, and =, the rules for transforming vector equations (not involving vector products) are the same as those for ordinary algebra. The student will now see the propriety of using these symbols with vectors. If the vectors were all parallel, the vector equations would become ordinary algebraic equations. Note that a numerical factor, as the 2 in 2**A**, affects only scalar part of the vector. Thus $2 \times 3_{10^\circ} = 6_{10^\circ}$.

Examples.—In the following examples take the values of **A**, **B**, and **C** to be those in Fig. (*a*) above, viz. 3_{15°, 8_{40°, 5_{160°.

1. Find the vector sums **A** + **B**, **B** + **C**, and **C** + **A**. *Ans.* 10.8_{33°, $7.05_{78.7^\circ}$, $3.1_{125.8^\circ}$. Add the three answers, and verify that $(\mathbf{A}+\mathbf{B})+(\mathbf{B}+\mathbf{C})+(\mathbf{C}+\mathbf{A})=2(\mathbf{A}+\mathbf{B}+\mathbf{C})=2\times 8.8_{61^\circ}=17.6_{61^\circ}$.
2. Find the vector differences **A** − **B**, **B** − **C**, **C** − **A**. *Ans.* 5.45_{234°, 11.35_{18°, 7.65_{173°. Add the three answers, and so verify that $(\mathbf{A}-\mathbf{B})+(\mathbf{B}-\mathbf{C})+(\mathbf{C}-\mathbf{A})=\mathbf{A}+\mathbf{B}+\mathbf{C}-\mathbf{A}-\mathbf{B}-\mathbf{C}=0$.
3. Obtain **A** − **B** and **B** − **A**. *Ans.* 5.45_{234°, 5.45_{54°.
4. Evaluate 3**A** − 2**B** + **C**. *Ans.* $10.38_{217.3^\circ}$.

130. Relative position and change of position regarded as a vector difference.—Let the positions of two places A and B be defined by the vectors $OA = \mathbf{A} = 8_{30°}$, say, and $OB = \mathbf{B} = 10_{85°}$. What is the position of B relatively to A? That is, how far and in what direction is B from A? If a person were at A, what change of position would bring him to B? Or if a person at A finds himself, after sundry wanderings, at B, what has been his change of position?

The answer to all these questions is the same, and in its simplest form, is the vector AB. Plotting A and B to a scale of $\frac{1}{8}''$ to 1 unit from the given data, we find that AB, which we shall call **C**, measures $8.5_{134.5°}$.

This answer would correspond to the direct route from A to B; but the change of position may be effected by any route. Suppose we select the route AO, OB. We might then say that the position of B relatively to A, or the change of position from A to B, is equal to $AO + OB$, that is, to $-\mathbf{A} + \mathbf{B}$, or $\mathbf{B} - \mathbf{A}$. Thus:—

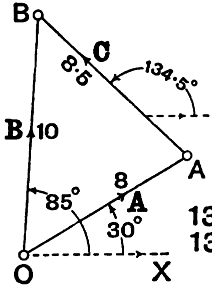
$$AB = \mathbf{C} = \mathbf{B} - \mathbf{A} = 8_{30°} - 10_{85°}.$$

131. Relative or apparent change of position.—Let points A and B have any initial positions A_1 and B_1. Let these be changed to A_2 and B_2, the changes being $A_1A_2 = \mathbf{A} = 8_{30°}$, $B_1B_2 = \mathbf{B} = 10_{85°}$, the same values as in Art. 130. What has been the change of position of B relatively to A? Or what new change in A's position would be required in order to bring A into the same position relatively to B as at the start?

Mark A_1 and B_1, and plot A_2, B_2 to scale from the given data. Draw B_2A_3 vectorally equal to B_1A_1. Then A_2A_3 is the relative or apparent change in B's position to a person moving with A. Calling A_2A_3 the vector **C** we have:—

$$A_2A_3 = A_2A_1 + A_1A_3, \text{ or } \mathbf{C} = -\mathbf{A} + \mathbf{B} = \mathbf{B} - \mathbf{A} = 8\;5_{134.5°}.$$

This is the same answer as before. And we see that the relative displacement is independent of the initial positions of A and B, and is the same as if both had started from any common point O. This vector difference may be interpreted thus:—If A had been stationary,



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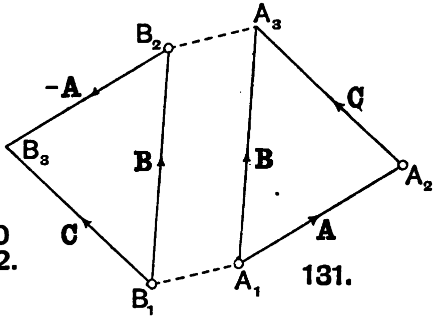
132.



131.

B's apparent movement would have been the actual displacement **B**; if B had been stationary, A's movement of **A** would have caused an apparent movement of B of − **A**. When both A and B move, the apparent motion of B is the sum of **B** and − **A**.

There are other useful ways of solving this problem, of which the following is one:—The relative positions of A_2 and B_2 would be unaltered by any displacement they might receive in common. Let this common change be − **A**, bringing A_2 back to A_1, and shifting B_2 to B_3, where $B_2B_3 = -\mathbf{A}$. The relative change of position we are seeking is thus the same as if A had remained at rest at A_1, and B had moved from B_1 to B_3. Now $B_1B_3 = B_1B_2 + B_2B_3 = \mathbf{B} - \mathbf{A}$ as before.

132. Relative velocity and change of velocity.—Let a point A have a velocity **A** of 8_{30°, and a second point B have at the same time a velocity **B** of 10_{85°; what is the velocity of B relatively to A?

By velocity is meant the change of position in unit time. These changes being **A** and **B**, the figure of Art. 130 is also a diagram of velocity. And **C**, = **B** − **A**, being the relative displacement which has occurred in unit time, also gives the relative velocity of B to A. If A and B were two ships, the velocity of B as it appeared to a person in A would be **B** − **A**. Or, again, the vector difference **B** − **A** would also give the change of velocity which had occurred in a body moving at one instant with a velocity **A**, and subsequently with a velocity **B**.

133. Vector differences. Summary.—We may profitably bring together the results of the preceding three articles. Remember that **A** and **B** are vectors defining situation, change of situation, or time rate of change of situation, *i.e.* velocity.

(*a*) *The situation* **B** *referred to the situation* **A** *is the vector difference* **B** – **A**.

(*b*) *If a point has a position* **A**, *and subsequently a position* **B**, *the change of position that has occurred in the interval is* **B** – **A**.

(*c*) *If two points receive displacements of* **A** *and* **B**, *the displacement of the latter referred to the former is* **B** – **A**. *It is the same as if both had started from a common point.*

(*d*) *If two points have velocities* **A** *and* **B**, *the velocity of the latter referred to the former is* **B** – **A**.

(*e*) *If a point has a velocity* **A**, *and subsequently a velocity* **B**, *the change of velocity that has occurred in the interval is* **B** – **A**.

We may restate the rule for finding **B** – **A**. See Fig. 133.

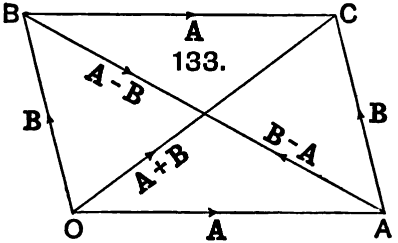
Rule.—*From any point* O *draw* O A *and* O B *to represent* **A** *and* **B**. *Join* A B. *Then* A B *represents* **B** – **A**, *and* B A *represents* **A** – **B**.

If we complete the parallelogram O C on O A, O B as sides, the fundamental theorem of vectors, stated in Art. 120, is that the diagonal O C represents the resultant or sum of **A** and **B**. We now see that the other diagonal represents the difference of **A** and **B**.

The equation **C** = **B** – **A**, where **C** denotes the motion **B** relatively to **A**, can be transformed into **B** = **A** + **C**. We may interpret this as follows :—The motion **B** is equal to the motion **A** together with the motion **B** relatively to **A**.

For instance, suppose a boy is walking on a moving raft; the motion of the boy is the one compounded of that of the raft, and the motion relatively to the raft.

134. PROBLEM.—**A cyclist goes at 10 miles per hour in a northerly direction. A south-west wind is blowing at**



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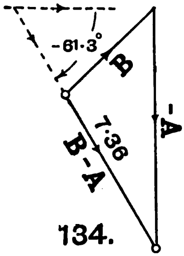


134.

5 miles per hour. Determine the wind which the rider experiences.

The velocity of the cyclist is $10_{90^\circ} = \mathbf{A}$ say.

The velocity of the wind is $5_{45^\circ} = \mathbf{B}$ say.

The velocity of the wind relatively to the cyclist $= \mathbf{B} - \mathbf{A} = 5_{45^\circ} - 10_{90^\circ} = 5_{45^\circ} + 10_{270^\circ} = 7.36_{-61.3^\circ}$ as measured, the scale of the diagram being $\frac{1}{8}''$ to 1 mile per hour.

That is, the wind is apparently one of 7.36 miles per hour *coming from* the quarter W. 61.3° N. or 61.3° north of west.

Instead of using the formula, we might reason from first principles.

Thus, if the atmosphere were still, the rider would, on account of his motion, produce a wind, as it were, against himself of 10 miles per hour *from* the north. The wind he experiences is the one compounded of the actual wind of 5 miles per hour from the south-west, and the self-created wind of 10 miles per hour from the north. This would give the same answer as above.

Or thus:—Add to both wind and rider such a velocity as would reduce the latter to rest, that is, a velocity of 10 miles per hour *towards* the south. The relative velocity of wind and rider is unchanged. The wind now is the one compounded of 5 miles per hour from the south-west, and 10 miles per hour from the north, and as the cyclist is motionless, this is the wind he experiences.

135. PROBLEM.—**Solve the vector equation**

$$24_{18^\circ} + 14_{84^\circ} + 32_{165^\circ} + A_{238^\circ} + B_{140^\circ} = 0.$$

The equation tells us that the resultant or sum of the five vectors is zero, in other words, that the five-sided vector polygon is **closed.** And we require to find the two magnitudes *A* and *B* so as to make this polygon close.

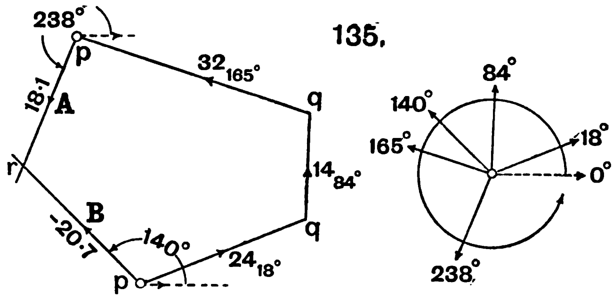
As a preliminary to drawing the polygon, it will be advisable to set out the five given directions, as shown to the right. This can be done very accurately by first drawing a circle of 10 units radius, and then employing a table of chords, making use of supplementary and complementary angles.

We begin the polygon by adding the three known vectors 24_{18°, 14_{84°, 32_{165° (scale 1 cm. to 10 units); this gives us the three sides lettered pqqp. We know, further, the directions of the two remaining sides **A** and **B**, *viz.*, 238° and 140°; so through p, p draw lines in these directions to intersect in r. The sides pr are the **closing sides** of the polygon. We must measure the lengths of these to scale in order to obtain the magnitudes *A* and *B*.

The length of **A** is 18.1 units, and the angle 238° makes the side **A** circuital with the sides first set out. Therefore $A = 18.1$. The length of **B** is 20.7 units, and the angle 140° gives a non-circuital direction to this side. That is, **B** is negative, and we must therefore write $B = -20.7$.

The solution of the equation is $A = 18.1$; $B = -20.7$.

Note.—The closing of the vector polygon always enables us to find two unknown quantities; in the present case, two magnitudes. If the magnitudes of **A** and **B** had been given, instead of their directions we could have determined the latter. When in any problem we find the resultant or equilibrant of a system of vectors, we determine one magnitude and one direction. A further case might arise in which the two unknown quantities were the magnitude of one vector and the direction of another. These different



135.

problems are all illustrated in Ex. 10 p. 140. In all these cases, after having drawn the sides which are completely given, it is seen that the closing of the polygon reduces simply to the drawing of a triangle from given conditions. And, as in Art. 45, there may be ambiguous cases in which more than one solution is possible. In the problem just solved there is only one solution.

136. Miscellaneous Examples.—

1. A ship at sea is sailing at 8.7 knots through the water apparently to the E., but there is an ocean current of 3.4 knots to the S.W. Find the velocity of the ship as regards the ocean bed. *Ans.* 6.75 E. 21° S. knots.
2. A ship A is sailing at 8.7 knots to the E., and a second ship B at 3.4 knots to the S.W. Find the velocity of B as it would appear to a person in A. *Ans.* 11.4 W. 12° S. knots.
3. Suppose the wind to be blowing at 5 knots from the N. Find the directions which the wind vanes would take, if carried by the three ships of examples 1 and 2. *Ans.* E. 22.5° N.; E. 30° N.; N. 42.7° W.
4. A steamer is going at 10 knots due north; the wind is blowing at 10 knots from the east; what is the direction of the trail of smoke from the steamer? *Ans.* S.W. If an instrument for measuring the velocity of wind were carried by the steamer, what would be the apparent wind registered? *Ans.* 14.14 knots from the N.E.
5. In Ex. 4, if the wind recorded by the instrument were apparently one of 20 knots from the N.E., the vessel's speed and course being unaltered, what would be the actual wind? *Ans.* 14.7 knots from the quarter E. 16.3° N.

6. A vessel is steaming eastwards. At 8 knots the trail of the smoke from the steamer goes S. 30° W. At 12 knots it goes S.W. Find the velocity of the wind. *Ans.* 9.7 S. 16° E. knots.

7. In Ex. 6, what would be the direction of the trail of the smoke if the speed of the vessel were 16 knots. *Ans.* W. 35° S.

Give the speeds of the wind as they would be felt by a person on board, for the three given speeds of the vessel. *Ans.* 10.8; 13.2; 16.3 knots.

8. Two railway trains, travelling in straight lines, are passing a crossing at the same instant; one, which we will call A, has a velocity of 920 yards per minute eastwards. The other, B, has a velocity of 745 yards per minute in the direction N.W. How far are they apart, and what is the direction from A to B at the end of one minute? *Ans.* 1540_{160° yards.

What is the velocity of B relatively to A? *Ans.* 1540_{160° yards per minute.

9. Suppose that during the next two minutes A turns round a curve of 20° to the left, and slows down to 700 yards per minute; and that B goes round a curve of 25° to the right, and increases in speed to 1000 yards per minute. Find the changes in the velocities of A and B. *Ans.* 355_{138°, $450_{65.8^\circ}$ yards per minute. Find also the relative velocity of B to A at the end of this time; and the change in relative velocity which has occurred during the interval. *Ans.* 1220_{145°, $475_{20.5^\circ}$ yards per minute.

10. Solve the following five vector equations:—

$A_\alpha = 10_{0^\circ} - 15_{30^\circ} + 30_{160^\circ}$. *Ans.* $A = 31.3$, $\alpha = 175^\circ$.

$A_\alpha + 10_{0^\circ} - 15_{30^\circ} + 30_{160^\circ} = 0$. *Ans.* $A = 31.3$, $\alpha = -5^\circ$.

$A_{60^\circ} + B_{310^\circ} + 10_0 - 15_{30^\circ} + 30_{160^\circ} = 0$. *Ans.* $A = 23.3$, $B = 30.1$.

$16_\alpha + 25_\beta + 10_0 - 15_{30^\circ} + 30_{160^\circ} = 0$. *Ans.* α and $\beta = 57^\circ$ and 335° or 47° and 325° or 313° and 35° or 303° and 25°.

$A_{140^\circ} + 24_\beta + 10_0 - 15_{30^\circ} + 30_{160^\circ} = 0$. *Ans.* $A = -9.7$, $\beta = 18.5^\circ$ or 8.5° or $A = -41.3$, $\beta = 101^\circ$ or 91°.

11. The following information is taken from the *Manchester Guardian* of August 7, 1902:—"Time of sunset 7 h. 56 m. Sunset is earlier for every ten miles south of Manchester by 32 sec., and for every ten miles east by 58 sec. Lamp time for cyclists is one hour after sunset."

Find the lighting up time on this day at Sheffield, which is 45° E. 10° S. miles from Manchester. *Ans.* 8 h. 51 m.

From any available map measure the position of York relative to Manchester, and find the lamp time at York on this day. *Ans.* 8 h. 53 m.

Find the direction of a line through Manchester such that the lamp time is the same on this day for all places on the line. Mention any places on, or very near the line. *Ans.* 29° E. of N. and W. of S. Keighley, Ripon.

12. Verify the following vector equations:—

$$A_{0^\circ} + A_{120^\circ} + A_{240^\circ} = 0.$$
$$A_{90^\circ} + A_{180^\circ} + A_{270^\circ} = A_{180^\circ}.$$
$$A_{0^\circ} + A_{60^\circ} - A_{120^\circ} = 2A_{0^\circ}.$$
$$A_{0^\circ} + A_{60^\circ} + 2A_{120^\circ} = 2.65A_{79^\circ}.$$

13. A yacht appears from its direction to be going due east, but the track of the water in the wake of the vessel shows that its motion through the water is 8° S. of E., and the log line indicates a speed of 10 knots. Find the component velocity in the direction of the vessel, and the lee-way perpendicular thereto. *Ans.* 9.9 knots east, 1.4 knots south.

CHAPTER XIII

UNIPLANAR CONCURRENT FORCES

137. Introduction.—We have now to deal with forces, such as we find exerted through human or animal agencies, or by the extension or compression of springs, gravitation, and in many other ways.

A force is essentially a directed quantity. When we hold an object such as a cricket-ball in the hand, we are conscious of a force action, the magnitude of which we know can be estimated in ounces, or pounds, etc., and the direction of which is vertically downwards, as can be ascertained experimentally with great precision by means of a plumb line.

The question now arises, is a force a vector? That is, being a directed quantity, does it obey the parallelogram law? Or, could two forces, without change of effect, be replaced by the single force represented by the diagonal of the vector parallelogram? The answer to this question must be an appeal to nature. We must examine the effects of the force actions we find around us, and see whether the law is true, and always true so far as we can test it.

Sir Isaac Newton, by a wonderful insight, and a close observation of natural phenomena, discovered the law, and was led to assert that force could be measured by the change of motion it produced in a free body on which it acted. Now change of motion is a vector, as we have seen when studying vector differences. So from Newton's law we infer that force is also a vector. But the student should experiment for himself, and when an opportunity occurs let him go straight to nature and obtain information first hand. Fortunately in this case the experiment is a very simple one, it is very instructive, and should be made personally by every individual student. The method of proceeding is described in the next article.

When we have verified that a force is a vector, we can then make use of the knowledge previously gained with respect to vectors in the last two chapters. Forces may be compounded or resolved by applying the parallelogram, or its equivalent the vector triangle, or its extension the vector polygon. And generally, the rules for adding vectors, which we have applied to linear motions, now become applicable in compounding and resolving forces.

138. Free vectors and localised vectors.—There is, however, one important difference between a displacement vector and a force vector which we must now point out.

In a linear displacement all points in a body are equally displaced. Two linear displacements, equal in magnitude and opposite in direction always cancel one another. Two elements, magnitude and direction, suffice to specify such a displacement. The line representative may have any position in space so long as it has the proper length and direction.

With regard to forces, however, the case is different. Two forces, equal in magnitude and direction, are not equivalent if they act in different lines. A pull at the handle of a drawer may open the latter when an equal and parallel pull at the side might fail. Two forces, equal in magnitude and opposite in direction, do not cancel one another except they act in the same line. They constitute a twisting action.

Three elements are necessary in specifying a force in a plane: magnitude, direction, and lateral position. A force vector has been called a vector localised in a line. The vector may be moved anywhere along the line backwards or forwards, but may not be moved sidewise. Previously we have been dealing with free or unlocalised vectors.

However, in this chapter, attention will be confined to concurrent forces, or forces whose lines of action all pass through the same point. This point being supposed known, two elements will be sufficient to define the force, as for a free vector. In drawing the force polygon, the vector lines will of course have to be shifted laterally in order to place them end to end. But when thus displaced they still represent the two elements of the force, magnitude and direction, and the closing side will give us these two elements of the resultant. And since in our case a point in the line of action is always known, the resultant will be completely known.

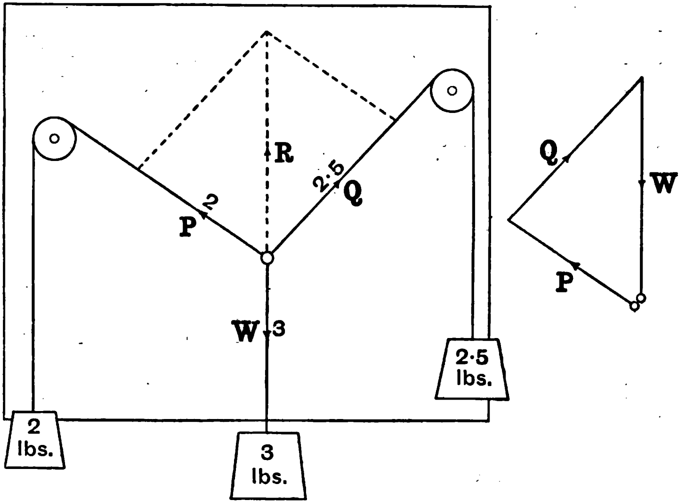
139. Experimental verification that a force is a vector.—In this experiment we require (*a*) a small ring to act as a point; (*b*) three flexible strings attached to the ring; (*c*) facilities for exerting steady pulls on the strings, and for varying these in magnitude and direction; and (*d*) means of measuring the magnitudes of the pulls and of recording the lines of the strings.

If three small spring balances are available, the experiment may be conducted as follows:—Place the ring with the attached strings on a sheet of drawing-paper on a horizontal drawing-board. Then fasten the other ends of the strings to the balances. Apply any three suitable pulls, and peg or clamp the outer ends of the balances to the board. Then mark on the paper the position of the ring and the lines of the strings, and read off the three pulls recorded by the index fingers of the spring balances.

An alternative plan, often adopted, is to fix the board vertically to a wall; to apply the forces by means of weights in scale-pans; and to vary the directions of the pulls by means of small guide-pulleys clamped to the board in any desired positions. The function of a guide-pulley is to alter the *direction* of the pull on the string without appreciably altering its magnitude.

A line diagram of the second form of apparatus is given in the figure, details of the construction of the pulleys, clamps, scale-pans, etc., being omitted. The weights in the three pans, together with the weights of the pans themselves, are supposed to be 2, 2.5, and 3 lbs., as indicated, and the lines taken by the strings when the oscillations have ceased, are supposed to be as shown. The ring is thus in equilibrium under the action of the three forces **P**, **Q**, **W**, of 2, 2.5, 3 lbs., acting along the three lines radiating from the ring.

Having detached the paper from the board, the verification of the vector law should now be made. Take any two of the forces, as **P** or **Q**; construct the parallelogram and obtain the diagonal **R**. Now examine whether **R**



represents a force equal in magnitude and opposite in direction to the third force **W**. If so **P** and **Q** might be replaced by **R** without destroying the equilibrium of the ring, and the parallelogram law would hold.

An alternative and equivalent construction would be to add the three lines representing **P**, **Q**, **W** as vectors, and to ascertain whether the vector triangle was closed, as ought to be the case, since the three forces balance. This triangle is drawn separately. It is represented as being very nearly but not quite closed. A slight discrepancy can be accounted for by inaccurate drawing, friction of the apparatus, etc.

The verification should be made several times for different weights and angles of string, so as to establish a strong presumption in favour of the law in cases like this. The experiment should then be continued as explained in the next article.

140. Experimental verification of the polygon of forces.—The law of the parallelogram of forces once established, that of the polygon of forces follows by deduction, as in Art. 120. It is, however, very desirable that the student should examine the case experimentally, in order to further verify the principle that forces are compounded by vector addition.

It is only required to provide several additional guide-pulleys, strings, and scale-pans. To note all the weights, and copy the lines of all the strings. Then to try whether the line representatives of all the pulls on the ring, when added as vectors, form a closed polygon, and if not, whether the failure to close is so slight that it can be accounted for by faulty apparatus or drawing.

141. Propositions relating to concurrent forces.—We now state in language having reference to concurrent forces, some properties which apply generally to vectors. In convincing himself of their truth, the student may, if it helps him, think of displacements instead of forces. If the statements are true for the former, they are also true for the latter, being true of vectors generally, subject of course to the distinction between free and localised vectors.

(a) *Two forces which are equal in magnitude, and which act in opposite directions along the same line, balance one another.* CONVERSELY. *Two forces which are in equilibrium must act in opposite directions along the same line.*

(b) *The resultant of two forces passes through their intersection, and its magnitude and direction may be obtained by constructing the parallelogram or triangle of forces, as described for vectors in Art.* 120.

(c) *Three concurrent forces will be in equilibrium if their magnitudes and directions can be represented by the circuital sides of a triangle of forces.* CONVERSELY:

(d) *In order that three forces may be in equilibrium they must be concurrent (unless they are parallel), and their magnitudes and directions must be represented by the circuital sides of a triangle of forces.*

(e) *The resultant of a concurrent system of forces is represented by a line through the common point, vectorally equal to the non-circuital closing side of a force polygon.* *CONVERSELY:*

(f) *It is necessary and sufficient for the equilibrium of a concurrent system of forces, that the force polygon shall close.*

(g) *If forces which balance one another be added to or subtracted from any system of forces, the resultant of the system is unaltered.*

(h) *If a system of forces is in equilibrium, any force reversed in direction is the resultant of the others.*

(i) *The algebraic sum of the components of a system of forces in any direction is equal to the component of the resultant in the same direction.*

This last proposition deserves to be stated more generally in terms of the projection (see Art. 123) of a vector polygon. The student should draw his own figure. Let ABCDE be a vector polygon, and abcde its projection on any line. Then for the polygon we have

$$\mathsf{AB} + \mathsf{BC} + \mathsf{CD} + \mathsf{DE} = \mathsf{AE},$$

and for its projection

$$\mathsf{ab} + \mathsf{bc} + \mathsf{cd} + \mathsf{de} = \mathsf{ae}.$$

Also for any portion of the polygon

$$\mathsf{BC} + \mathsf{CD} = \mathsf{BD},$$

and for its projection

$$\mathsf{bc} + \mathsf{cd} = \mathsf{bd}.$$

Examples.—The following vectors represent forces in lbs. acting at a point :—840_{0°, 1000_{58°, 240_{260°, 740_{-23°.

1. Find their resultant. *Ans.* 2000_{10° lbs.
2. Find the components in directions 0° and 90°. *Ans.* 1970, 348 lbs.
3. Find the magnitudes of two balancing forces whose directions are 70° and 170°. *Ans.* -705, 1750 lbs.
4. Find the directions of two balancing forces whose magnitudes are each 1500. *Ans.* 122° and 218°, or 142° and 238°.
5. Find the equilibrant or the force which would balance the system. *Ans.* 2000_{190° lbs.

We now give some problems worked out in detail.

142. Problem.—**Forces of 5 and 2 units act along lines OM, ON, whose directions make 70° with one another. Find the resultant.**

First method.—By the parallelogram of forces.

To any convenient scale, mark off along OM and ON, the lengths OP and OQ of 5 and 2 units.

Complete the parallelogram by drawing PR, QR parallel to OQ, OP. Draw the diagonal OR, and direct it from O to R.

Then OR represents the required resultant. It measures 6 units and makes 18.5° with OM and 51.5° with ON.

Second method.—By the triangle of forces.

Draw any line ab in the same direction as OM and 5 units long. From b draw bc in the same direction as ON and 2 units long. Join ac and direct it from a to c, *i.e.* non-circuitally.

From O draw OR *vectorally equal* to ac, that is, having the same magnitude and direction. Then OR represents the resultant as before.

Note.—The scale of the figure is $\frac{1}{4}$ inch to 1 unit.

143. Problem.—**A given force is represented by OR. Find its components along the given lines OM, ON.**

Draw RQ and RP parallel to MO and NO. Direct OP and OQ outwards from O, as OR is directed.

Then OP and OQ represent the required components.

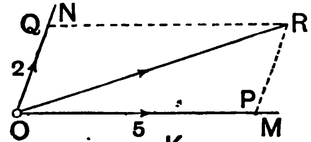
144. Problem.—**A force of 34 lbs. acts along HK. Find its component along LL which makes 35° with HK.**

Rectangular component is to be understood.

Select a convenient scale, say $\frac{1}{4}''$ to 10 lbs., and mark off HK 34 units long. Direct it from H to K.

Project HK on LL, obtaining hk. Direct the latter from h to k. Then hk is the required component; it measures 27.9 lbs.

Or by calculation, hk = HK cos 35° = 34 × 0.819 lbs. = 27.9 lbs. Find the other component. *Ans.* 19.5 lbs.



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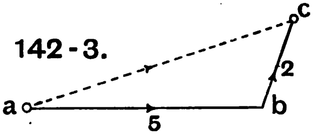

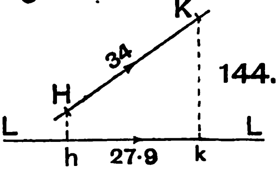

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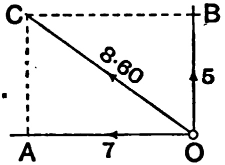

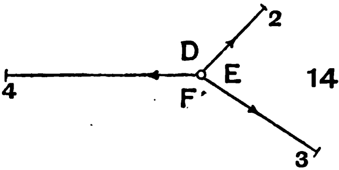

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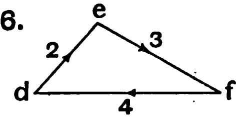


145. Problem.—**The rectangular components of a force are 7 and 5 lbs. Find the force.**

Draw any two perpendicular lines, along which, to a convenient scale, mark off from their intersection O, OA and OB to represent 7 and 5 lbs.

Complete the rectangle on OA, OB, and draw the diagonal OC. Then OC represents the force; it measures 8.60 lbs., and makes 35.5° with OA and 54.5° with OB.

Or by calculation, $OC=\sqrt{OA^2+OB^2}=\sqrt{49+25}=\sqrt{74}=8.60$ lbs. Tan $AOC=AC/OA=5/7=0.714$. $\therefore$ $AOC=35.5°$.

146. Problem.—**A small ring is at rest under pulls of 2, 3, and 4 lbs. exerted through strings. Find the angles between the strings.**

Draw a triangle of forces def with its sides 2, 3, and 4 units long and circuitally directed.

From the centre of the ring draw lines *vectorally equal* to the sides of the triangle.

These represent the three pulls, and the angles D, E, F between them measure 133.4°, 75.6°, and 151°.

147. PROBLEM.—**A load of 2 tons is suspended from a pin A, which is maintained in position by the pull of a horizontal tie AL, and thrust of a strut MA inclined at 25° to the horizontal. Find the pull and thrust.**

Draw ab 2 units long and directed vertically downwards to represent the given load. From the ends a and b draw two lines parallel to the tie and strut to intersect in c, and direct the sides bc, ca circuitally with ab.

Then abc is the triangle of forces representing the equilibrium of the pin A. The pull of the tie is found by measuring ca to be 4.28 tons. And the thrust of the strut MA is 4.72 tons, as given by scaling bc.

148. PROBLEM.—**Four given pulls and thrusts K, L, M, N act on a point as defined in the figure. Find their equilibrant.**

Construct a vector polygon thus:—Select a convenient scale (say $\frac{1}{4}''$ to 10 units), and from any point a draw ab, bc, cd, de respectively parallel, and similarly directed to **K, L, M, N,** and 72, 40, 105, and 58 units long.

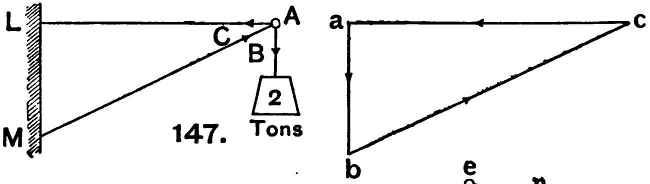
Join ea and direct it circuitally, *i.e.* from e to a. A line **R** drawn through the common point, vectorally equal to ea, represents the required balancing force.

Measuring ea, the magnitude of **R** is found to be 88.5 units. The angles E and A measure 45° and 121°.

Observe the two systems of notation. In one, the forces are denoted by **K, L, M, N, R,** and the corresponding sides of the force polygon by **k, l, m, n, r.**

In the other, known as **Bow's Notation,** the *angles* round the point are lettered A, B, C, D, E, and the corresponding *corners* of the force polygon a, b, c, d, e. In this system a force is referred to by naming the two letters in the angular spaces on each side of it. Thus the force BC (or CB) would mean the force **L,** and the corresponding letters b, c would appear at the ends of the side **l.**

On going round the point (clockwise), the sequence A, B, C, D, E of the angles is seen to be the same as the circuital sequence a, b, c, d, e of the corners of the vector polygon. If we agree that the two letters which denote any force shall be named in the sequence derived from the rotation, *e.g.* BC for **L,** then the sequence bc agrees with the circuital arrow.



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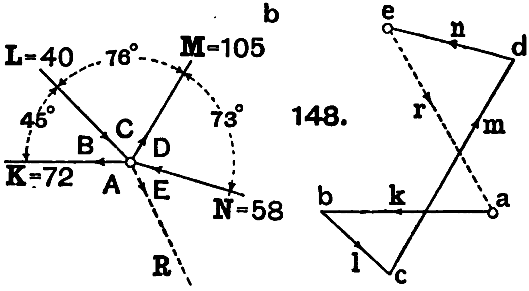

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Examples.—**1.** Find the force whose horizontal and vertical components are 72.2 lbs. and 45.6 lbs. *Ans.* 85.3 lbs. inclined at 32.3° to the horizontal.

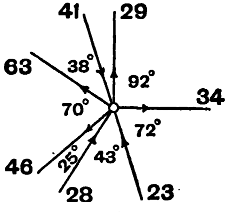
2. A force of 100 lbs. acts at an angle of 53.7° with the horizontal; find its horizontal and vertical components. *Ans.* 59.2 lbs.; 80.6 lbs.

3. Two forces of 3.5 and 6.7 lbs. act in directions which include an angle of 75°; find their resultant. *Ans.* 8.32 lbs. making 24° with the larger force and 51° with the smaller.

4. The lines of action of two forces include an angle of 120°; one of the forces is 10 lbs., and their resultant is 9 lbs. Find the other force. *Ans.* 7.4 lbs. or 2.62 lbs.

5. A horizontal string 8 feet long is attached to two walls, and from a point 3 feet from one end a load is hung which is increased until at 10 lbs. the string breaks, the deflection when this happens being 6″. If a piece of the same string were to hang vertically, what load would it carry? *Ans.* 37.9 lbs.

6. Determine the resultant of the concurrent system of forces defined by the figure. *Ans.* A pull of 54.4 making 20.4° with the 41 force, and 17.6° with the 63° force.



149. PROBLEM.—**Forces which act at a point are in equilibrium; all except two, M and N, are known completely, but only the lines of action of these latter are given. Find the magnitudes and senses of M and N.**

This problem is solved by applying theorem (*f*), Art. 140.

We shall again illustrate Bow's notation, and to be able to apply it the figure has been first prepared by drawing the lines of action of the forces each with one end at the common point, so that there shall be the same number of angular spaces as there are forces. It has also been arranged that **M** and **N** shall be adjacent so as to bound one of the spaces; this is always possible since a force may be represented either as a *pull* from one side or as an equal *push* from the other.

The spaces are then labelled A, B, C, D, X. Clockwise rotation being adopted, the construction may be described as follows:—

Draw ab, bc, cd the three sides of the force polygon corresponding to the three given forces AB, BC, CD.

Close the polygon by drawing the two remaining sides dx, ax parallel to DX, AX, intersecting in x.

The forces **M** and **N** are then given by the sides dx, xa taken circuitally, or as determined by the clockwise sequence DX, XA. **M** is seen to be a pull, and **N** a thrust.

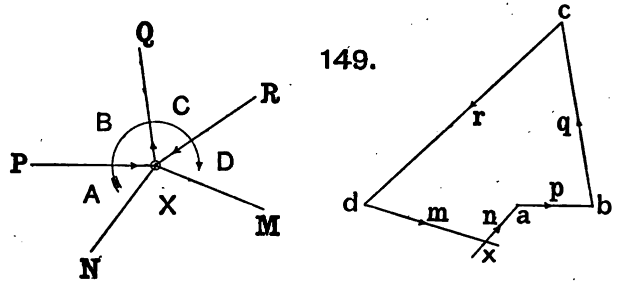
150. PROBLEM.—**A load W is suspended from a jib crane; to find the thrust in the jib CA and the pull in the tie AB.**

First consider the equilibrium of the tie. At the end A it is in contact with the pin, and between the two surfaces there is a force action consisting of equal and opposite forces, one on the pin, the other on the bar. If the joint is supposed frictionless, the line of action of these forces must pass through the centre of the pin.

A similar force occurs at the other end B of the tie. Neglecting its weight, these are the only forces acting on the bar, since the latter is nowhere else in contact with anything. And as the two forces balance they must be equal and opposite. The bar must therefore be subject to a *direct pull* or a *direct thrust.* The tie is shown separately, subject to the action of the two forces **T**, **T**, indicating a pull.

Similarly for the jib. And generally, *in a hinged frame, loaded at the joints only, the forces at the ends of any bar must be equal and opposite, and must act along the line joining the centres of the pins.*

Equilibrium of the pins. To obtain these forces and to distinguish



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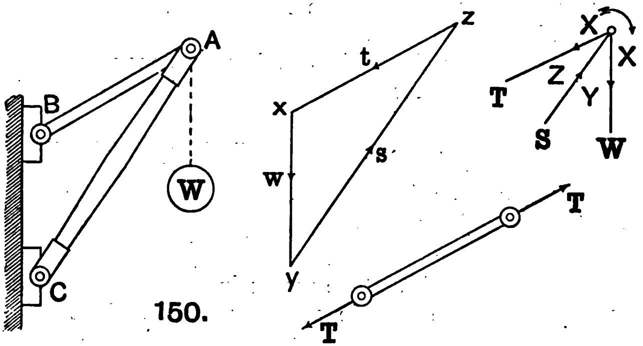

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pulls from thrusts, we must draw the closed polygons of forces for the pins which hold the frame together at the joints.

Consider the forces on the pin A. There is the vertical force **W**, being the load suspended from it. There is the force which the tie exerts on it, acting in the line of the tie. And there is the force exerted by the jib along the line of the jib. The pin is shown separately, with the lines of action of these three forces drawn. The angular spaces round the pin are labelled X, Y, Z.

To draw the triangle of forces for the pin A. Adopting clockwise rotation, the sequence of letters for **W**, the known force is XY. So set off xy vertically downwards to represent **W**. Draw xz, yz parallel to XZ, YZ. By watch-hand rotation about the pin we obtain the sequences YZ, ZX for jib and tie, and we direct these lines near the pin A to agree with the sequences yz, zx of the triangle of forces.

YZ or **S** is directed towards the pin A, indicating a *thrust* from the jib; and ZX or **T** is directed away from the pin, denoting a *pull* from the tie. The magnitudes are obtained by measuring yz and zx.

151. PROBLEM.—**A hinged triangular frame is in equilibrium under given forces P, Q, R applied at its corners. To find the forces in the three bars of the frame.**

Since the forces **P, Q, R** maintain equilibrium, their lines must all pass through one point (not shown), and their magnitudes and senses must be given by the circuital sides of the triangle of forces abc.

To employ Bow's notation, the lines of the external or applied forces are all drawn *outside* the frame, and with their ends at the joints. The external and internal spaces are then lettered A, B, C, D.

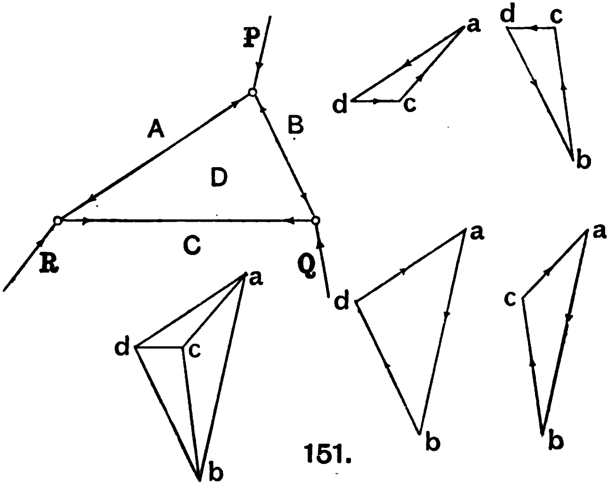
To obtain the forces in the bars, we may draw the triangles of forces for the three joint pins. Clockwise rotation is adopted.

The upper joint. The spaces round this joint, named in watch-hand sequence, are A, B, D. The triangle of forces named circuitally is abd. The forces BD, DA, directed in the senses bd, da, both act towards the joint, and indicate thrusts from the two inclined bars.

The left-hand joint. The spaces in clockwise sequence are C, A, D. The triangle of forces is cad. The forces AD, DC, directed as ad, dc, act the first towards, the second away from the joint. The horizontal bar of the frame is thus in tension.

The right-hand joint. The spaces are B, C, D, and the triangle of forces is bcd.

The force diagram.—The four triangles of forces may be superposed on one another so as to form the figure abcd. The six lines of the latter measured to scale give the magnitudes of **P, Q, R** and of the forces in the bars. It is called the *force diagram* for the frame. The four triangles need not have been drawn separately, and are only given here for purposes of illustration; they are all included in the force diagram. The arrowheads would conflict if put on the lines of the force diagram, and they are purposely omitted; but the sense of any force may be obtained by the con-



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vention of Bow's notation as previously explained, rotation about the joints of the frame being always clockwise.

152. Miscellaneous Examples.

1. Draw six lines oa, ob, oc, od, oe, of radiating at equal angular intervals of 60°. These lines are respectively the lines of action of forces of 40, 100, 70, 60, 100 and 30 lbs., all acting towards o (pushes), except those along oa and of, which are pulls. Determine and state the magnitude and direction of the resultant of the forces. *Ans.* A pull of 173 lbs. making 30° with oa and also 30° with of.
2. If **R** is the resultant of two forces **P** and **Q**, we have the vector equation $\mathbf{P}+\mathbf{Q}=\mathbf{R}$, which can be transformed into $\mathbf{P}=\mathbf{R}-\mathbf{Q}$ or $\mathbf{Q}=\mathbf{R}-\mathbf{P}$. Thus one force is the vector difference between the resultant and the other force. *Example*—Find **Q** if $\mathbf{P}=14_{0^\circ}$ lbs. and $\mathbf{R}=24_{8^\circ}$ lbs. *Ans.* $12.6_{173.3^\circ}$ lbs.
3. Three ropes are attached to a ring O. When three boys are pulling their hardest at the ropes, the angles between the latter are measured and found to be AOB = 132°, BOC = 118°, COA = 110°. Which is the strongest boy and what per cent stronger is he than the weakest boy? *Ans.* B is 25 per cent stronger than C and 7 per cent stronger than A.

4. What would have been the angles between the ropes in the last question, if **A** and **C** had exerted equal pulls and **B** a pull 20 per cent greater? *Ans.* **AOC** = 106°, **AOB** = **BOC** = 127°.

***5.** The directions and magnitudes in lbs. of five unequal forces acting at a point **a** are given. Determine the direction and magnitude of their resultant. (1884)

***6.** Five given forces act as shown at a point. Determine by construction two forces acting along the given dotted lines which will keep the point in equilibrium. Write down the magnitudes and indicate the directions of these forces. Use a scale of 1" to 100 lbs. (1887)

***7.** A load of 5 tons is suspended from **a**, the apex of a triangle formed of two bars **ba**, **ca** fixed in a vertical wall. Determine and write down the forces in the bars **ba**, **ca**. (1893)

***8.** A hinged frame is given resting on horizontal supports. The given loads of 260 lbs. and 380 lbs., and the supporting forces **P** and **Q** are all vertical. Draw the force diagram for the frame to a scale of 1" to 100 lbs. Measure and tabulate the forces in all the bars, and the reactions **P** and **Q**. *Ans.* (partial) **P** = 235 lbs. **Q** = 405 lbs.

***9.** The figure shows a braced cantilever loaded at the joints. Draw the force diagram to a scale of $\frac{1}{4}''$ to 1 cwt. Measure and tabulate the forces in the four bars **p**, **q**, **r**, **s** of the frame. Distinguish ties from struts by the signs + and − respectively. *Ans.* See the table.

Bar.	p	q	r	s
Force—cwts.	8.35	−6.7	−14.3	19.9

10. A rope is stretched between and secured to two opposite points on the walls of a room 15 feet apart. A man weighing 11 stone 6 lbs. grasps the rope at the middle of its length and raises himself from the ground, causing the rope to sag 11 inches. Find the tension in the rope due to the man, and the horizontal force tending to pull the walls together. *Ans.* 659 lbs., 654 lbs.

11. A horizontal telegraph wire attached to the top of a pole, is deflected 80° at the post, so that the two branches of the wire make 100° with each other. The tension of the wire on both sides of the post is 200 lbs. Find the horizontal force tending to overturn the pole. *Ans.* 306 lbs.

This force is resisted by the horizontal component of the tension in a wire stay, which passes from the top of the pole to

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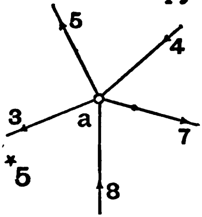

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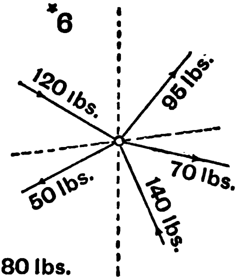

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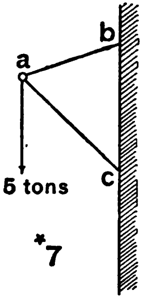

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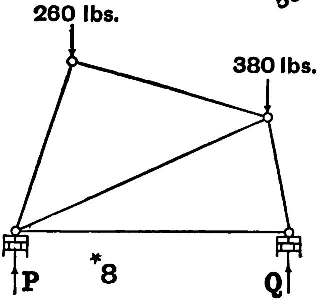

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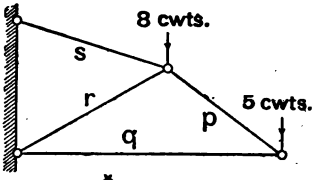


*9

the ground at an inclination of 60° to the horizontal. Find the tension in this stay. *Ans.* 613 lbs.

12. A person drawing a truck exerts a force of 45 lbs. in a direction making 35° with the direction of motion. What is the effective force on the truck urging it forwards? *Ans.* 45 cos 35° or 36.9 lbs.

What is the non-effective or side force? *Ans.* 45 sin 35° or 25.8 lbs.

What is the vector sum of these two component forces? *Ans.* 45 lbs., making 35° with the direction of motion.

CHAPTER XIV

GEOMETRY OF THE POINT IN SPACE

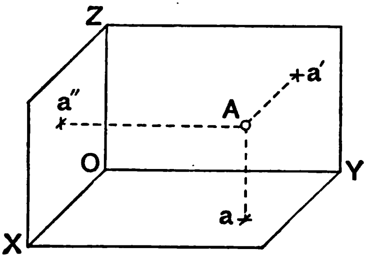
153. Introduction.—We now begin the subject of the geometry of three-dimensional space.

Our attention has hitherto been confined to the geometry of the plane, which has only two dimensions. In previous problems, the lines and figures, having been restricted to a plane, we have been able to take this as the plane of the drawing-paper, and to set out the shapes of the figures directly.

But now a third dimension is introduced, and our drawing-paper will no longer contain all the points and lines with which we shall have to deal. Some or all may be situated outside, above or below the paper.

Yet the positions of points, lines and planes in space, and the forms of solids, possessing length, breadth, and thickness, must somehow be exhibited on a sheet of drawing paper possessing only length and breadth. And we must be able to make quantitative measurements of lengths and angles connected with such figures in space. Our subject will now consist mainly in the development of ways and methods of effecting these objects by means of *scale drawings*.

154. The position of a point in space defined by rectangular co-ordinates.—We have seen in Art. 56 how the position of a point *in a plane* may be defined by reference to two perpendicular *lines*. In like manner the position of a point *in space* may be defined by reference to three



mutually perpendicular *planes*. Thus, the situation of a small object in a room is known if we know its height above the floor and its distances from two adjacent walls.

Let the **planes of reference** be as shown in the figure, one *horizontal*, and the other two *vertical* and at right angles to each other, the latter being distinguished as the *front* and *side* vertical planes. These planes intersect in three lines, also mutually perpendicular, called the co-ordinate axes, one being vertical, and the other two horizontal. The point common to the three axes and the three planes is called the *origin*, and is denoted by the letter O, the three axes being labelled respectively OX, OY, OZ, the latter being the vertical one. The three planes of reference may also be designated as the planes of XY, YZ, ZX respectively.

The position of any point A may now be defined by its perpendicular distances a′A, a″A, aA from the three planes of reference. These three distances are called the **rectangular co-ordinates** of the point, and may be denoted by x, y, z. The co-ordinate x is the distance of A from the plane YZ, measured parallel to the axis OX. Similarly for the other two: y is measured parallel to OY and z parallel to OZ. Thus x and y are the horizontal co-ordinates, and z is the vertical co-ordinate.

The point whose co-ordinates are x, y, z may for shortness be referred to as the "point (x, y, z)."

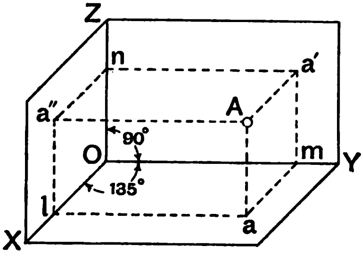
155. The position of a point in space shown by pictorial projection.—The specification of the rectangular co-ordinates of a point is a numerical method of defining its position. But this is not sufficient for our purpose. In descriptive geometry it is further necessary to set out this position in a scale drawing from which the lengths of the co-ordinates can be measured.

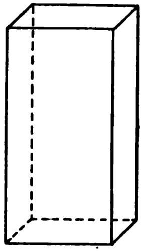
As the drawing is confined to one plane, and the figure which the drawing represents is *not* plane, some convention is necessary in order to connect the two, and so render the signification of the drawing precise and definite.

Our previous illustration, which we have here reproduced with additions, shows one very useful way of doing this. In setting out this figure we first draw OZ vertical, then OY in any other direction, say horizontally to the right, and OZ in any third direction, as by using the 45° set-square. These lines represent the axes, and we now further agree that lines in space parallel to the axes shall be represented on the drawing by lines always parallel to OX, OY, OZ respectively.

We next choose *any* three scales that may prove to be convenient for setting off the lengths of lines which are parallel to the axes, taking care to use these same scales when measuring the diagram. The simplest plan would be to use identical scales for all three sets of lines, but in order to avoid an undue appearance of distortion, it is better to adopt, for lengths parallel to OX, a scale half the size of that used for the other two axes. Our subsequent pictorial views will be set out on this understanding, so that dimensions may be readily measured from them.

In the accompanying figure, the scale for the directions OY, OZ is ¼″ to 1″, and that for the direction OX is therefore ⅛″ to 1″. Using these scales, the co-ordinate x is seen to be 3″, as can be ascertained by measuring any of the four equal lines a′A, OI, ma, na″. The co-ordinates y and z measure 4″ and 2″, as the student should verify from the figure.





In plotting the figure, observe that, starting from O, the point A may be reached by setting off the co-ordinates in any order. If the order were x, y, z, A would be reached by the path Ol = 3″, la = 4″, aA = 2″. The alternative route Om, ma′, a′A would correspond to the order y, z, x, and so for any other sequence.

A drawing of a building brick is shown to the right, the scales being ⅛th, ⅛th, and $\frac{1}{16}$th. Measure the size of the brick. *Ans.* 9″ by 4½″ by 3″.

Examples.—**1.** Draw a pictorial view of the planes of projection. Let OX be 3½″ (to scale), OY = 4½″, and OZ = 2½″. Let the scales be full size for the directions OY, OZ, and half size for the direction OX. In this figure plot the point (3″, 4″, 2″).

2. Using the same scales represent a cube of 2″ side.

3. Draw a pictorial projection of the building brick, in some position other than the one shown above. Scales ¼ and ⅛th.

156. Explanations of terms.—If from any point A in space, a straight line or **projector** be drawn to meet a plane in a, the point A is said to be **projected** on the plane, and the point a is called the **projection** of A. If the projector Aa be perpendicular to the plane the projection is known as **orthogonal** or **orthographic.** Otherwise it is **oblique.** When projection is spoken of without qualification, orthographic projection is always to be understood.

Thus **the projection of a point on a plane is the foot of the perpendicular let fall from the point to the plane.**

Notation.—We shall use a capital letter for a point in space, and the same small letter for its projection.

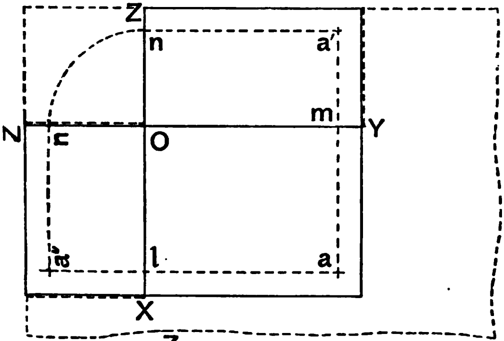
157. The position of a point in space exhibited by orthographic projection.—To illustrate this method, which is one often adopted, let us refer for a moment to the pictorial view on the last page. The points a, a′, a″ are the orthographic projections of A on the planes of reference, or **planes of projection**, as they may now be called. The point a is the **plan** of A. The other two projections, a′ and a″, are **elevations**. To distinguish, we shall call YZ the **front** vertical plane, and a′ the **front** elevation. And ZX is the **side** vertical plane and a″ the **side** elevation.

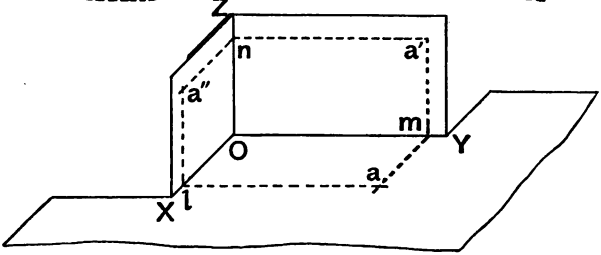
These three projections are now to be used to exhibit the position of A. It is necessary that they be all brought into one plane, so that they may be set out on the drawing-paper. To effect this we must, as a preliminary step, suppose the point A and the perpendiculars removed. We then imagine the two vertical planes to be turned **backwards** through an angle of 90°, so as to bring them both into the horizontal plane of XY.

The result, when this operation has been completed, is shown by the upper figure opposite (the dotted outline excepted). This figure exhibits the position of A graphically by means of three projections, a, a′, a″. The scale is the same for all lines of the diagram, being $\frac{1}{4}$ full size in this case. If we wish to obtain x from the figure, we may measure either ma, Ol, or na″. Any one of la, Om, or na′ would give us y. And there are four lines all equal to z.

To further illustrate the meaning and significance of this projection, let the student make a model in the following manner. First set out the figure at the top left-hand corner of the drawing-paper as indicated by the dotted lines (use the $\frac{1}{4}''$ scale for measuring the dimensions). Then cut the paper where the double lines appear, and indent it along the lines OY, XOZ. The two vertical planes of projection may now be turned into their true positions in space, as shown in the pictorial view below. The actual point A in space may be left to the imagination.

Note.—On comparing the two figures, the student should





observe that as regards the front vertical plane YZ, the lower figure is an exact copy of the upper one. Thus in the pictorial projection all figures in the plane YZ will appear in their *actual shape*. In the other two planes there is, of course, pictorial distortion, though, as previously explained, dimensions can be measured from them.

Example.—Draw the plan abcd of a rectangular room ABCD, length, AB or CD, 16 feet; breadth, BC or AD, 12 feet. Scale $\frac{1}{4}$ inch to 1 foot.

Show the plan p of a point P inside the room, distant 4 ft. and 7 ft. from the walls AB and BC. Measure the distances of P from the four vertical corners where the walls meet. *Ans.* PA=9.85′, PB=8.05′, PC=10.61′, PD=12.02′.

If the height of P above the floor is 8 ft., what are the distances of P from the four corners a, b, c, d of the floor? *Ans.* Pa=12.7′, Pb=11.33′, Pc=13.3′, Pd=14.45′.

158. A point in space given by two projections.—The student may have noticed that the three projections of the last article are rather prodigal of information. He will see that the co-ordinates of the point A can be measured from any two of the three projections. One may therefore be dispensed with. This is usually the side elevation, the plan and front elevation only being retained, as shown in the accompanying figure.

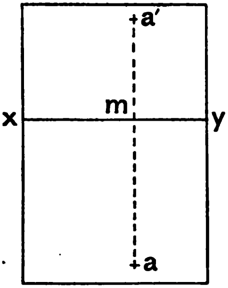
The point O and the lines OX, OZ are omitted, and the axis OY is now generally lettered xy, and is known as the **ground line**, or the xy line, or simply as xy. The z and x co-ordinates are referred to as *height above the horizontal plane* and *distance in front of the vertical plane.*

The figure should make all this quite clear. The scales are the same as before, viz. $\frac{1}{4}$ for the plan and elevation, and $\frac{1}{4}$ and $\frac{1}{8}$th for the pictorial view.

Example.—Measure the position of the point A from both plan and elevation, and pictorial view. *Ans.* 2″ above the horizontal plane and 3″ in front of the vertical plane.

159. A point in space represented by a figured plan.—A further simplification is still possible, leading to a method of representation which is often very useful. When the plan a of a point A is drawn, two elements defining the position of A are known. We can measure the x and y co-ordinates whenever we choose, and from any axes we may wish to assign. There remains only the height or z co-ordinate lacking, and the missing information can be supplied, without the necessity of drawing an elevation, by simply writing the height on the plan, say as a suffix. Thus in the figure, $a_{2''}$ indicates that the point A is situated vertically above its plan a, at a height of 2″, as illustrated by the pictorial view, which is set out to the same scales as before.

The reader who is acquainted with maps contoured at different levels, will recognise an example in which figured plans are used to delineate the shape of the irregular surface of the ground.



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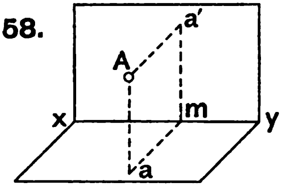

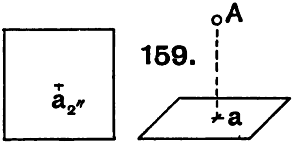

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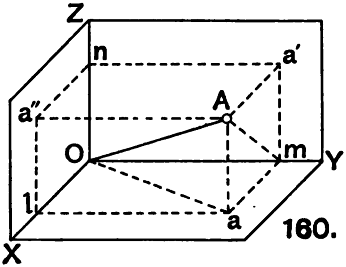


160.

160. Problems on the position of a point.—We now state some problems, and indicate methods of solution. It will be seen that these all reduce to solving right-angled triangles. The triangles may be drawn anywhere on the paper, detached from the projections if preferred.

(a) To find the distance of A from the origin.—Set out the true shape of the right-angled AOa, and measure its hypotenuse OA. Verify that $OA^2=x^2+y^2+z^2$.

(b) To find the inclination of OA to the horizontal plane.—Measure the base angle AOa of the triangle just drawn.

(c) To find the angle which OA makes with the vertical OZ. Measure the angle AOa of the same right-angled triangle. Observe that the angles of (b) and (c) are complementary.

(d) To find the distance of A from OY or from xy.—Construct the right-angled triangle Aam, and measure its hypotenuse Am.

(e) To find the angle which OA makes with OY.—Construct the right-angled triangle Aom, and measure its base angle Aom.

(f) To draw the three projections of OA. These are the lines Oa, Oa′, Oa″.

161. A point in space defined by polar co-ordinates.—We now give our last method of defining the position of a point. It is one with which the student should be acquainted.

For reference, instead of three planes, we choose (1) a fixed point; (2) a fixed line through the point; (3) a fixed plane through the line. These are usually taken as (1) the point O; (2) the line OZ, vertically upwards; (3) the plane XOZ.

To define the position of any point A we then proceed as follows:—

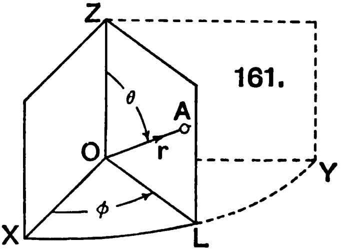
First.—Conceive a plane, initially coinciding with the plane XZ, to turn about OZ as axis, like a door swinging on its hinges, until it contains the point A. Let ϕ be the angle turned through, or the angle through which the door would have to be opened, *i.e.* the angle XOL.

Next.—Imagine a line, starting from the position OZ, to be turned in this plane about O as centre, like a rod being moved in the plane of the door, until it contains the point A. Let θ be the angle turned through by the line or rod, *i.e.* the angle ZOA.

Third.—Suppose a point starting from O to travel along the line, like a ring being slid on the rod, until it reaches A. Let r be the distance moved through by the point or ring, *i.e.* the length OA.

Then these three quantities, ϕ, θ, r define the position of A. They are known as polar or vector co-ordinates. The meaning should be clear from the pictorial projection without further description. The scales for this view are the same as before, viz. $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{8}$th.

As an illustration, consider how the position of a place A on the earth's surface is defined by latitude and longitude. The point O corresponds to the centre of the earth. OZ is the north polar axis. XOZ is the meridian plane through Greenwich. XOY is the plane of the equator. The angle ϕ is the east *longitude.* The angle LOA is the north *latitude*, so that θ is the complement of the latitude,



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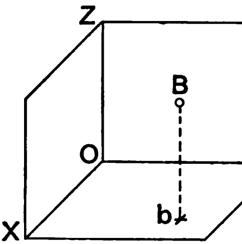


Ex. 1.

or the *co-latitude* as it is generally called. The radius of the earth, being practically constant, is not specified, but would correspond to the polar distance r.

Thus in speaking of polar co-ordinates ϕ, θ, r we may appropriately call them the longitude, co-latitude, and polar distance.

If the x, y, z co-ordinates are given, the ϕ, θ, r co-ordinates are readily found either by construction or by calculation. In regard to construction, see the appropriate cases, Art. 160.

As to calculation, the student may verify that

$$r^2 = x^2 + y^2 + z^2; \quad \cos\theta = \frac{z}{r}; \quad \tan\phi = \frac{y}{x}.$$

162. Miscellaneous Examples.

***1.** A pictorial projection is given above of the planes of projection and of a point B in space. Scales, $\frac{1}{4}$ size for directions OY, OZ; $\frac{1}{8}$ size for direction OX. Use the 45° set-square for OX.

- **(a)** Measure from the diagram (by the aid of one construction line) the x, y, z co-ordinates of B. *Ans.* 3.5″, 2.7″, 2.4″.
- **(b)** Copy the figure four times the size shown, the scales thus becoming full and half size.
- **(c)** Draw the three projections of B as in Art. 157, near the top left-hand corner of the drawing-paper, in a position suitable for constructing the model. Cut out this model. (Take O 3″ from the top and left edges of the paper.)
- **(d)** Find the distance of B from the origin, both by construction and calculation. *Ans.* 5.02″.

(e) Find the inclination of OB to the horizontal plane. *Ans.* 28.6°.

(f) Find the angle BOZ. *Ans.* 61.4°.

(g) Find the distance of B from OY. *Ans.* 4.24″.

(h) Find the angle BOY. *Ans.* 57.6°.

(k) Draw the three projections of OB. Measure their lengths. *Ans.* Ob = 4.32″, Ob′ = 3.61″, Ob″ = 4.24″.

2. Taking the same data as in Ex. 1, represent the point B by its plan and elevation, as in Art. 158. Cut out three sides of the vertical plane so as to be able to turn it about xy into its upright position. Draw the pictorial view for this case. Determine and measure the distance of B from the ground-line, xy. *Ans.* 4.10″.

3. Represent the point B, Ex. 1, by a figured plan as in Art. 159. Draw the pictorial projection as there shown.

4. Find the polar co-ordinates of B, Ex. 1. *Ans.* $r=4.97''$; $\theta=61.4^\circ$; $\phi=40.4^\circ$.

5. Copy Fig. 161 three times the size shown. Then let the scales be full and half size. Now measure the x, y, z co-ordinates of A. *Ans.* 2.62″, 2.17″, 1.35″. Also determine the polar co-ordinates r, θ, ϕ. *Ans.* 3.66″, 68.4°, 39.6°.

*6. A point C is given by its plan and elevation. Scale $\frac{1}{2}$. Measure the distance of C from the horizontal and vertical planes of projection, and from the ground line. *Ans.* 1.30″, 0.78″, 1.52″. Represent this point by a pictorial projection.

*7. A pictorial view is given showing the horizontal plane and a point D in space. Scales $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{8}$th. Represent this point by a figured plan. What is the height of the point? *Ans.* 2.31″.

*8. The figured plans of the ends E, F of a line in space are given. Scale, full size. Find the true length of EF. Determine the inclination of EF to the horizontal plane. Draw an elevation of EF on a vertical plane defined by the given ground line xy. *Ans.* 2.24″, 42.1°.

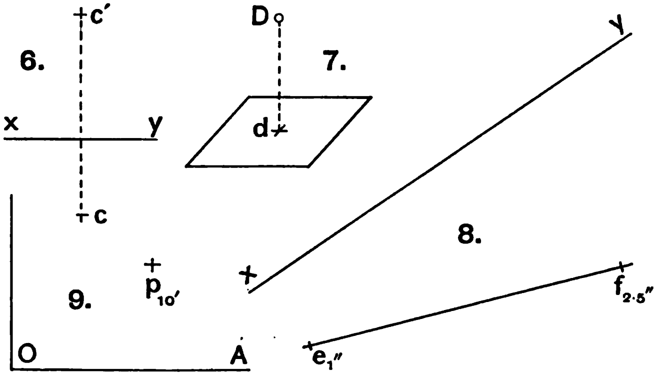
*9. The diagram shows the plan of a portion of a room, drawn to a scale $\frac{1}{2}$ inch to 10 feet. O is one corner of the floor. P is a point above the floor given by its figured plan $p_{10'}$ (indexed in feet).

Determine and measure in feet the distance of P from the corner O of the floor. *Ans.* 20.8 feet.

Find also and measure the distance of P from the line OA on the floor. *Ans.* 14.8 feet.

10. Draw the plans and elevations of the following points :—

G, situated 2″ above the horizontal plane and 1.5″ from the vertical plane.



H, situated $2.5''$ above the horizontal plane and in the vertical plane.

K, situated in the horizontal plane and $1''$ from the vertical plane.

L, situated in both planes of projection.

In each case find the distance of the point from the ground line. *Ans.* $2.5''$; $2.5''$; $1''$; $0.0''$.

11. The rectangular co-ordinates x, y, z of a point M are $3'$, $4'$, and $5'$. What are the polar co-ordinates r, θ, ϕ? *Ans.* $7'$, 45°, 53°.

12. The polar co-ordinates r, θ, ϕ of a point N are $3''$, 60°, 45°. What are the x, y, z co-ordinates? *Ans.* $1.84''$, $1.84''$, $1.5''$.

CHAPTER XV

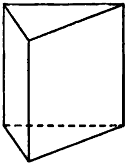
PROJECTIONS OF SIMPLE FIGURES

163. Preliminary explanations.—It may appear somewhat illogical to take up the projection of solids before that of lines and planes of which they are made up. But experience shows that the first ideas of projection are more surely conveyed to a student, and more readily understood and assimilated by him, when presented in connection with something more tangible than the line and plane, something which will better assist the imagination in picturing clearly the space relations involved. For this purpose simple geometrical plane and solid figures answer admirably, and some of the fundamental forms of the latter with technical terms used in connection therewith must first be explained.

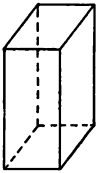
Just as a point in motion traces out a line, so a moving line generates a surface. A plane surface, *i.e.* a plane, may be conceived as having been swept out by a straight line moving parallel to itself and always intersecting a fixed straight line, or otherwise as being guided by two straight lines which intersect or are parallel.

The intersection of a line and plane is a point. The intersection of two planes is a straight line. That of three planes a point, viz. the point where the line of intersection of any two cuts the third. Parallel planes are such as never intersect, though produced.

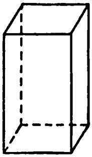
We are only concerned here with **polyhedra** (*many-faced*), which are solids bounded entirely by plane figures,



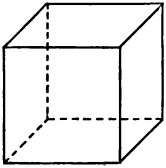
Triangular Prism



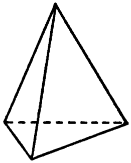
Rectangular Prism



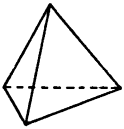
Square Prism



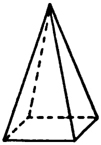
Cube



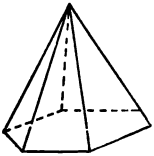
Triangular Pyramid



Regular Tetrahedron



Square Pyramid



Hexagonal Pyramid

called **faces**, meeting in straight lines called **edges**, and in points called **corners**. And we shall only require two forms of these, *prisms* and *pyramids*.

In a **right prism**, two faces, known as the **ends** or **bases**, are parallel, and the remaining faces are all perpendicular to the planes of these two. The **axis** is the imaginary line joining the centres of the ends, and its length is the **length** or **height** of the prism. The shape of the ends gives the name to the prism, as *triangular*, *square*, etc. See the figures. A **cube** is a prism having **six square faces**.

In a **pyramid**, the **base** is any polygon, and the other faces are triangles having a common **vertex** or **apex**. The imaginary line joining the vertex to the centre of the base is the **axis**. If the axis is perpendicular to the base we have a **right pyramid**, otherwise the pyramid is **oblique**. The perpendicular distance from the vertex to the base is the **altitude** or **height**. A **regular tetrahedron** has four faces, all equilateral triangles.

164. Projections of an equilateral triangle.—We now show the **plans and elevations** of an equilateral triangle of 2″ side, vertex A, base B B, in a number of different positions lettered (**a**) to (**f**). The scale is ½.

The student will be assisted by using a paper model of the triangle, which may have a folding margin at the base so that it will stand upright. When considering the plans let the reader think of himself as looking vertically downwards on the model. As regards the elevations, he may consider that his position is now changed so that he is facing the vertical plane, and viewing the model horizontally.

(**a**) The triangle is standing upright on its base, with its plane parallel to the vertical plane. First draw the elevation, which is an equilateral triangle, 2″ side. Then project the plan. The plan is an edge or a profile view of the triangle.

(**b**) The triangle is still upright, but it has been turned through 45° about a vertical axis. Copy the plan (**a**) in the new position (**b**). Then project the elevation.

(**c**) The triangle is turned through a further angle of 45°, so that its plane is now perpendicular to both planes of projection. Both plan and elevation are now profile views.

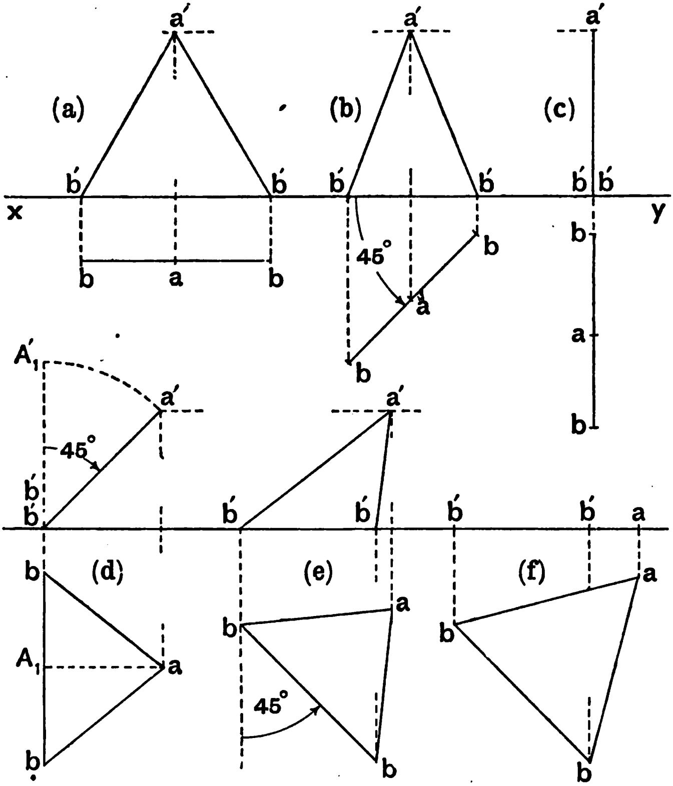
(**d**) This position is derived from (**c**) by tilting the triangle through 45° about the base. Copy the elevation (**c**) in the new position (**d**). Then project the plan. Observe that the path of the vertex is a circular arc, which has A_1a' for its elevation and A_1a for its plan.

(**e**) Suppose the triangle when in position (**d**) to be rotated through 45° about a *vertical axis.* The plan will now be differently placed as regards xy, but *the shape of the plan is unaltered, and the height of the vertex is the same.* So copy the plan (**d**) in the new position (**e**) and then project the elevation.

(**f**) If the triangle, situated as in (**e**) be turned into the horizontal plane about its base, we shall obtain (**f**). First draw the plan, an equilateral triangle, 2″ side. Then project the elevation.

164a. Projections of a square.—The lower figure shows, ¼ size, the projections of a square, 2″ side, obtained exactly as in the last case. Let the student observe that parallel and equal sides of the square have always parallel and equal projections.

Example.—Set out the above projections of the triangle and square, full size, but take 30° instead of 45° in each case.


a′
(a)
b′
b′
b
a
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x
a′
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y
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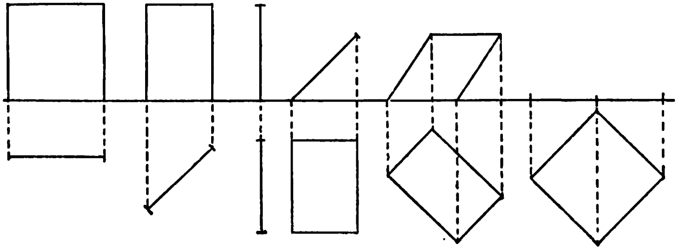


165. Projections of a cube.—The method of the two last articles is continued, the subject now being a cube of $2\frac{1}{2}''$ edge. Scale $\frac{1}{4}$.

A **model** of the cube may be made in paper by cutting out the shape of a **development** of its surface, as shown at D, $\frac{1}{8}$ size. After being indented along the lines, the paper can be folded into the cubical shape. The margins indicated are for glue or seccotine.

The **pictorial projection.**—This is shown at p. 171, quarter size. The axes and scale ratios are the same as those adopted in the last chapter, being 45° and $\frac{1}{2}$ for the foreshortened edges.

Plans and elevations.—Four different positions of the cube are represented at (a), (b), (c), (d), the third of which is further illustrated by a pictorial view (c) drawn to scale.

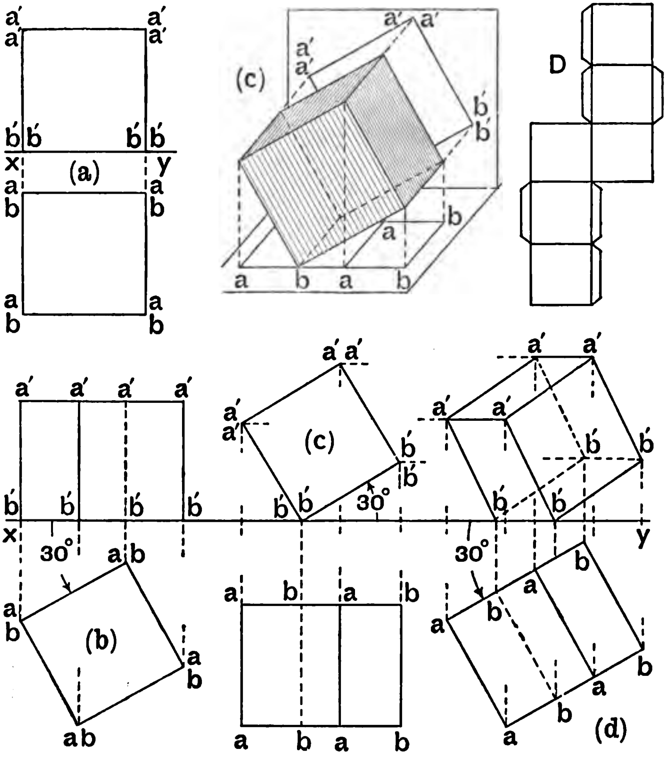
(a) The cube rests with a face on the ground plane, its vertical faces being respectively parallel and perpendicular to the vertical plane of projection. We have lettered the four upper corners each A, and the four lower corners B.

In *plan*, the vertical edges AB appear as points; the four vertical side faces project into lines; and the upper and lower faces overlap and coincide in this view. Corresponding remarks apply to the elevation.

(b) The cube still rests on a face, but it has been turned through 30° about a vertical axis. First draw the plan; then project the elevation.

(c) This position is obtained from that of (a) by means of a tilt of 30° about a lower edge perpendicular to xy. Copy the elevation (a) in the new position (c). Then project the plan. The pictorial view above illustrates how the cube is situated in this case.

(d) Without altering the tilt of 30°, let the cube in position (c) be rotated through 30° about a vertical axis. As in Art. 164 (e), *the shape of the plan, and the heights of the corners are unaltered.* Therefore copy the plan (c) in the position (d). Then project the elevation, as shown by the vertical and horizontal dotted lines in the diagram.



Examples.—1. Draw, full size, the four projections of the cube above, taking angles of 45° instead of 30°. Draw also the pictorial view for your position (**b**).

2. Draw the plan and elevation of a rectangular prism 9″ by $4\frac{1}{2}$″ by 3″ :—(**a**) when resting with a large face on the ground, with the long edges parallel to **xy**, scale $\frac{1}{4}$; (**b**) after a tilt of 45° about a $4\frac{1}{2}$″ edge ; (**c**) after a further rotation of 45° about a vertical axis. Draw the pictorial view for position (**c**), taking the usual axes and scale ratios.

166. Projections of a triangular prism.—Length 3″; ends, equilateral triangles 2″ side. Scale 1 cm. to 1″.

Development and model.—A development is shown at D. This should be set out full size, and a model of the prism made in paper.

(a) **Projections on three rectangular planes.**—The prism rests with a rectangular face on the plane XY, with its ends parallel to the plane XZ.

First draw the **plan**, a rectangle 3″ by 2″ with the line **aa** representing the top edge. The **end elevation** is the equilateral triangle shown. The **front elevation** is a rectangle, the height of which is obtained from the end elevation.

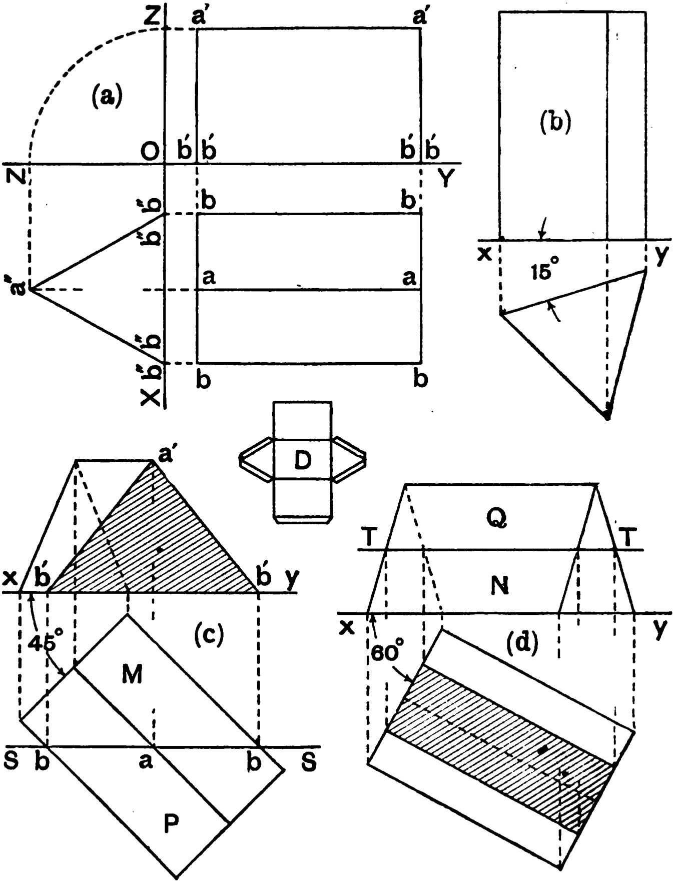
(b) Here the prism stands on one end, a vertical face making 15° with the vertical plane. First draw the plan, then project the elevation.

(c) **Sectional elevation.**—The solid rests on a rectangular face, its ends at 45° to the vertical plane. Draw the plan, and project the elevation of the *complete* solid, obtaining the height from (a).

Now draw a line SS, bisecting the axis and parallel to xy. Let this be the plan of a **vertical section plane,** cutting the prism into two parts, P and M. Now suppose the portion P in *front* of SS to be removed, thus exposing the section, which is the triangle ABB. Obtain by projection the elevation a′b′b′ of this triangle, and erase the elevation of P, leaving only that of M. Indicate the section by **hatched** or **section lines,** on the **sectional elevation.**

(d) **Sectional plan.**—Draw the rectangular plan in the position shown, and project the elevation.

Draw a horizontal line TT, at say half the height of the upper edge. Let this be the elevation of a **horizontal section plane,** dividing the prism into two parts, Q and N. Suppose the upper part Q to be removed, leaving N, and revealing the section, which is a rectangle. Determine the plan of this rectangle by projecting from the elevation, and



distinguish the section by hatched lining on the **sectional plan.** This plan is to be drawn as if Q were removed.

Note.—Sectional projections are largely employed by engineering and architectural draughtsmen in order to show the internal arrangements and construction of things.

167. Projections of a pyramid.—The pyramid is defined by the given indexed plan vabcd, the figured suffixes indicating heights. Scale ½. V is the vertex, and ABCD the base.

From an inspection of the plan, we can make several inferences. First, we see that the edge VA lies in the horizontal plane, since the heights of V and A are both zero. Next, as the plan of the base is a parallelogram, the base itself must also be a parallelogram, since *parallel lines have parallel projections, and conversely.* Lastly, we can find the height of C; for there is the same vertical rise from B to C as from A to D, *viz.* 1.8″. The height of C is thus 0.8″ + 1.8″ = 2.6″.

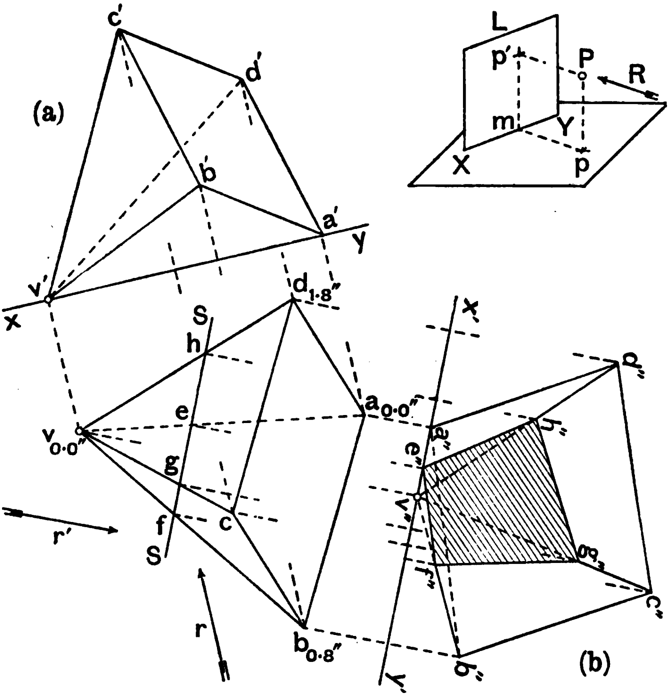
(a) Draw an elevation of the pyramid on any ground line xy.—By this is meant an elevation on any *vertical plane* meeting the horizontal plane in xy.

We must be prepared to draw not only a front and side elevation, but also an elevation on *any* vertical plane. That is, we must be able to show the appearance of an object when viewed horizontally from the N., S., S.W., or any other direction. Thus let P be any point, R the direction of view, L a vertical plane perpendicular thereto, xy its ground line. p′ is the elevation of P on this plane. On the drawing-paper the two planes of projection must coincide. This may be conceived as having been brought about by turning L backwards about xy, or, what comes to the same thing, by turning the horizontal plane down into the vertical plane about the same line. Observe that when this has been effected, pmp′ will be a straight line (or projector) perpendicular to xy, and that mp′ will be equal to the height pP. We thus have the following rule for drawing the elevation on *any* xy:—

Rule.—*Project the points in plan perpendicular to* xy, *and on the projectors mark off, for the elevations, distances above* xy *equal to the heights of the points above the ground.*

The elevation (a) is obtained in this manner, as the student can easily verify by measurement. This elevation shows the appearance of the pyramid when viewed horizontally in the direction r.

(b) Draw a sectional elevation of the frustum on SS.—Take a vertical plane of projection x′y′ parallel to the vertical section plane SS. On x′y′ draw the elevation v″a″b″c″d″ of the pyramid. The above rule is again applied.



The section.—The *plan* is **efgh**. For the *elevation*—Since H is in the edge VD, its elevation h″ must be in v″d″, so project from h to h″ as shown. Proceed similarly for the other points E, F, G.

The line r′ shows the direction of view. As in Art. 166 (c), the portion of the pyramid in front of SS must be supposed to be removed when drawing the sectional elevation, thus uncovering the section, and leaving only the frustum of the pyramid behind the section plane.

168. PROBLEM.—**A stone cross is represented in plan and elevation. Draw the pictorial projection.**

The diagram needs little explanation. As previously observed, in accordance with our choice of axes and scales, figures which are parallel to the front vertical plane are reproduced without pictorial distortion.

So in the pictorial view, the front face of the cross is an exact copy of the elevation. And the foreshortened edges, representing the thickness, are drawn in the direction 45°, and of length equal to half the width of the plan, *i.e.* to a scale of ½, in agreement with our general understanding with the reader on this matter.

169. Miscellaneous Examples.

***1.** The plan is given of an oblique prism or parallelepiped, the heights of four of the corners being indexed. Calculate the heights of the other corners, and index their plans. Draw and complete the plan of a section of the prism by a plane which cuts three edges in points A, B, C, whose plans are given.

2. The linear scale for Fig. 1 is half size. Copy the plan full size. Then draw an elevation on any ground line you like to select, and determine the plan of the section of the prism made by a horizontal plane 1.6″ high.

3. Draw a pictorial view of the cross, Fig. 168, supposing the cross to be lying flat on its face instead of standing upright. Scale, double size.

4. A triangular prism, length 3″, has equilateral ends, 2″ side. Draw its plan when resting on one side of the base, with the plane of the base inclined to the horizontal plane at 60°.

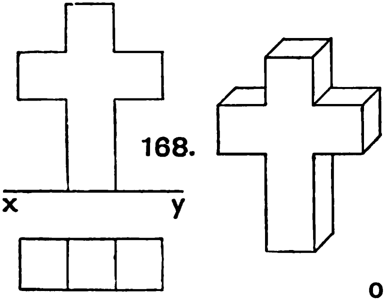
5. Copy the plan of the pyramid, Art. 167, full size. Then draw an elevation on any ground line different to those shown. And determine the plan of the section of the pyramid made by a horizontal plane 1.5″ high.

6. An equilateral triangle of 2″ side rests with one side on the ground, its plane being inclined at 45°. Draw the figured plan and an edge elevation. Draw also an elevation on an xy making 45° with the first xy.

7. Draw the plan and elevation of the cross, Fig. 168, after it has been tilted about an edge of the base so as to bring an edge of the crosspiece on to the ground.

8. Index the plans of the three given points A, B, C, Fig. 1. *Ans.* $a_{1.27''}$, $b_{2.33''}$, $c_{1.12''}$.

9. Copy the plan of the pyramid, Art. 167, full size. Then draw an



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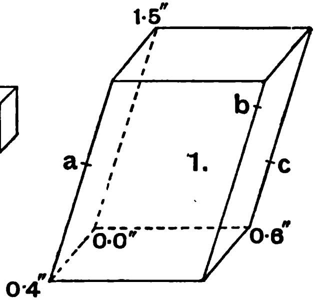


elevation on a ground line perpendicular to the plan of VA. Now determine the plan of the pyramid after the solid has been turned about VA until the face VAB rests on the ground.

CHAPTER XVI

THE STRAIGHT LINE IN SPACE

170. The methods of representing a line in space.—The simplest way of defining the position of a line is to define the positions of two points in it. Thus the problem of representing a line reduces to that of representing two points. We have already explained several methods of representing points; these apply to lines. Thus:—

(**a**) The position of a line AB may be defined by a numerical statement of the rectangular co-ordinates of A and B. As, the line (1″, 1″, 1″), (3″, 3½″, 2″).

(**b**) The position may be shown by a pictorial view.

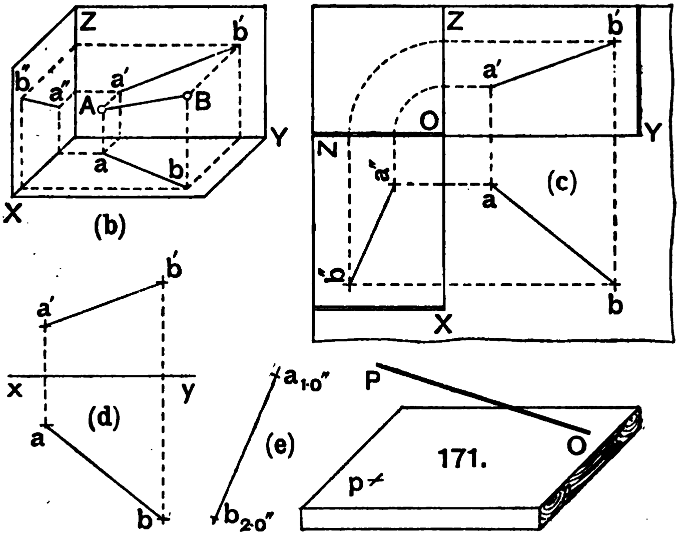
(**c**) The position of AB may be exhibited by three projections on the planes of reference.

(**d**) The line may be represented by a plan and elevation.

(**e**) The position may be given by a figured plan.

The line AB of (**a**) is represented in all four ways (**b**), (**c**), (**d**), and (**e**) in the figure. Scale ¼. In (**c**) the projections are shown conveniently placed on the paper for the model. See Art. 157.

171. The angle between a line and plane; meaning of, and how measured.—An excellent plan of illustrating this is to provide a model in wood, say 6″ square and ½″ thick for the plane, with a stout wire rod firmly inserted in an oblique position to represent the line. No construction lines whatever should be shown, not even the plan **p**, and the



171.

model should be painted black, like a blackboard, so that lines can be drawn in chalk by the student himself, and then erased.

The model would be given to the student, and he would be asked :—

(a) To measure the length of the wire and the angle between the wire and plane.

(b) To take measurements of the model, and from his dimensions (without further reference to the model) to draw a pictorial projection to scale, like Fig. 171. And

(c) Supposing the plane of the model to be horizontal, he should represent the line by a figured plan, and also by a plan and elevation; and by construction from these he should determine the length and inclination of the line, and compare the results with his previous direct measurements of the same in (a).

The procedure would be somewhat as follows:—

(**a**) The angle would be measured by the swivelling blade of the clinograph, or roughly by the open legs of the dividers. Without any explanation being required, the student would naturally adjust these in position **directly underneath** the wire, and by this operation would answer the question as to what is meant by the angle between a line and plane. In order to adjust the clinograph properly, and secure good results, he would be led to draw, in chalk, the projection of the wire on the plane. To do this he must first find the projection p of the free end P of the wire. This would require the use of two set-squares in conjunction, or of one set-square in two different positions. By this operation again the student is acquiring useful knowledge; he has discovered the important fact that *a line is perpendicular to a plane, when it is perpendicular to* **two** *intersecting lines in the plane.*

The student thus finds out for himself that the inclination of a line to the horizontal plane is the *angle between the line and its plan.*

The model might now be placed with its plane vertical, representing the front, side, or any vertical plane. In this position the model teaches him that the angle between a line and any vertical plane is *the angle between the line and its elevation on the plane.*

Or generally, suppose the model to be held with its plane in any inclined position, we have an illustration of the following general rule:

RULE.—**The angle between a line and plane is measured by the angle between the line and its projection on the plane.**

Calculation.—If AB be the line, ab its projection on any plane, and θ the angle between line and plane, note that—

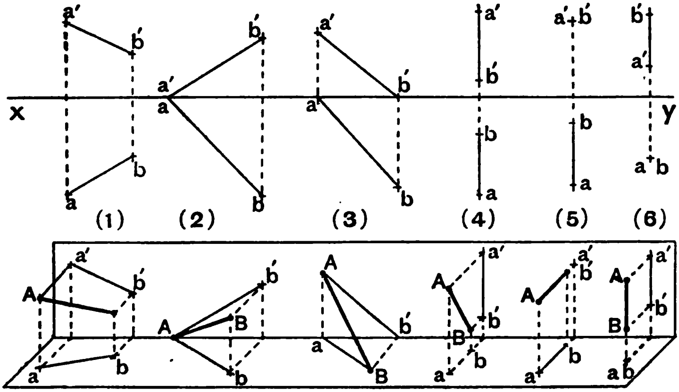
$$ab = AB \times \cos\theta, \qquad \text{and } \cos\theta = ab \div AB.$$

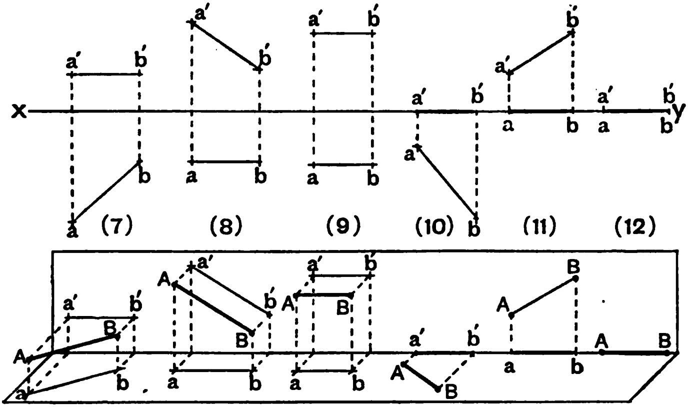
That is, we can calculate the length of the projection by *multiplying the length of the line by the cosine of the angle between line and plane.* And we can calculate the cosine of the angle by *dividing the length of the projection by the length of the line.*

(**b**) In taking dimensions for the pictorial view, the student has virtually to make use of rectangular co-ordinates in locating the ends of the wire.

(**c**) The student should invent his own methods of finding the length and inclination of OP from his projections.

Example.—If a model is not available, take Fig. 171 as a poor substitute, and find the length and inclination of OP. The figure is drawn $\frac{1}{4}$ size. *Ans.* 5.0″, 28.3°. Find also the lengths of Op and Pp, and by calculation verify that Op = OP cos POp; Pp = OP sin Pop.





172. Lines in various positions.—Twelve different cases are shown in plan and elevation, accompanied by pictorial views. In the first four the line is inclined to both planes of projection, and both plan and elevation are foreshortened. In the others the line is parallel to or lying in one or both planes.

173. PROBLEM.—**A line AB is given by its figured plan; find the length and inclination of the line. Scale $\frac{1}{2}$.**

First method. By a right-angled triangle.—Construct a right-angled triangle, the base of which is equal to the plan ab, and the height equal to the *difference* in height of A and B $=(1.5'' - 0.7'') = 0.8''$. Then the hypotenuse is equal to required length of AB, and the base angle gives the required inclination.

Second method. By a rabattement.—From the given plans a and b draw perpendiculars to ab, on which mark off lengths aA_0, bB_0 equal to the given indexed heights.

Then $A_0 B_0$ is the required length of AB; it measures 2.25″. And the angle which $A_0 B_0$ makes with ab, viz. 22°, is the required inclination.

Note.—If the student refer to p. 183, Fig. (b), he will see that the second construction represents the quadrilateral ABba when turned into the horizontal plane about its base ab as axis. A figure turned into a plane of projection is said to be **rabatted.**

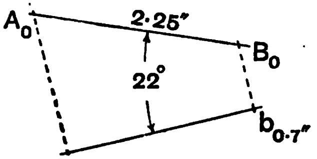
174. PROBLEM.—**A line AB is given by a plan and elevation. Find the length of AB and the angles which the line makes with the planes of projection. Scale $\frac{1}{2}$.**

If the line is parallel to either plane of projection, the length and angles are exhibited directly by the projections, and only require measurement. See Figs. (5) to (12) p. 185. We need only consider the general case.

First method. By right-angled triangles.—We leave this to the reader as an exercise.

Second method. By rabattements.—At a and b erect perpendiculars to ab, and mark off lengths aA_0, bB_0 equal to the heights ma′, nb′. The required length is $A_0 B_0$; it measures 2.52″. The required inclination to the horizontal plane measures 12° as shown.

Next, at a′ and b′ draw perpendiculars to a′ b′, on which mark off $a'A_1$, $b'B_1$ equal to the distances ma, nb. Then $A_1 B_1$ gives the length a second time; and the angle between $A_1 B_1$ and a′b′, measuring 21°, give the inclination of AB to the vertical plane.



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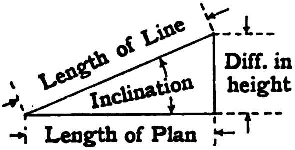

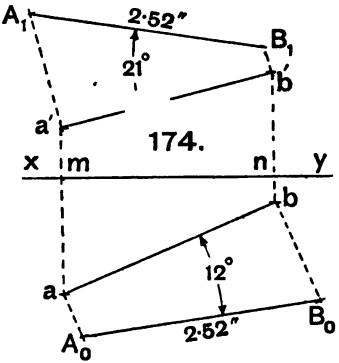

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Note.—We have here two rabattements of **AB**. First into the horizontal plane about **ab**; next into the vertical plane about **a′ b′**. Refer, if necessary, to the pictorial view, p. 185. The student may cut his drawing-paper so as to allow the planes of projection, and the two rabattements to be turned into their true relative positions in space.

175. PROBLEM.—**The rectangular co-ordinates $(x_1\,y_1\,z_1)$, $(x_2\,y_2\,z_2)$ of the ends of a line are (2.5″, 1.5″, 0.5″) and (0.5, 4″, 2″). Calculate the length of the line and the angles which the line makes with the planes of reference.**

We leave it to the student to verify the following calculations :—

Length of line $= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$= \sqrt{(0.5-2.5)^2+(4-1.5)^2+(2-0.5)^2}$

$= \sqrt{4+6.25+2.25} = \sqrt{12.5} = 3.53''$.

Length of plan $= \sqrt{(x_2-x_1)^2-(y_2-y_1)^2} = \sqrt{4+6.25} = 3.20''$.

Inclination to horizontal plane

$= \cos^{-1}3.20/3.53 = \cos^{-1}0.906 = 25^\circ$ (p. 24).

Length of front elevation

$= \sqrt{(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{6.25+2.25} = 2.91''$.

Inclination to plane **YZ**

$= \cos^{-1}2.91/3.53 = \cos^{-1}0.823 = 34.6^\circ$.

Length of side elevation

$= \sqrt{(z_2-z_1)^2+(x_2-x_1)^2} = \sqrt{2.25+4} = 2.50''$.

Inclination to plane **ZX**

$= \cos^{-1}2.50/3.53 = \cos^{-1}0.710 = 44.7^\circ$.

176. Application of the right-angled triangle.—As the reader may already perceive, and as he will find fully confirmed later, problems in solid geometry involving quantitative measurements, often reduce to problems on the solution of right-angled triangles.

Thus in the right-angled triangle OPM let the hypotenuse OP be the length of any line, and let the base OM be the length of its plan; then the base angle POM is equal to the inclination of the line to the horizontal plane, and the perpendicular PM is equal to the *difference* in the vertical heights of its ends.

Or in the right-angled triangle OPN let ON be the length of the elevation, then the angle PON is the angle which the line makes with the vertical plane, and PN is the *difference* in the distances of the ends of the line from the vertical plane.

The triangle can be solved when any two independent elements are given, and we have seen in Art. 42 that nine cases may arise. So in the present application, the problems admit of considerable variety in the data, and yet are all solved by means of triangles.

Let the student work the following examples, either by construction, or by calculation, or preferably by both methods :—

Examples.—**1.** The length of the plan of a line is 2″; one end of the line is in the ground, and the other end is 1″ high. Find the length and inclination of the line. *Ans.* 2.23″, 26.5°.

2. A line is 3″ long; its plan measures 1.8″; find its inclination to the ground, and the height of one end above the other. *Ans.* 31°, 1.55″.

3. A line 2.5″ long is inclined at 50°, the lower end being 0.6″ high. Find the length of its plan and the height of the upper end, and represent the line by a figured plan. *Ans.* 1.61″, 2.51″.

4. A line 2.7″ long makes 35° with the vertical plane, what is the length of its elevation? *Ans.* 2.22″. If the line is moved so that its elevation measures 1.22″, what angle does it now make with the vertical plane? *Ans.* 63°. If in both cases, the nearer end of the line is 1″ from the vertical plane, how far is the other end from the vertical plane? *Ans.* 2.56″, 3.41″.

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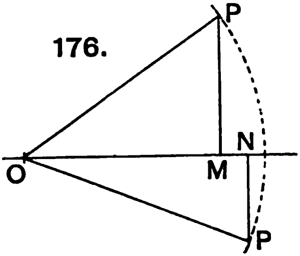

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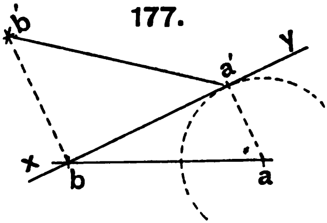


5. A line AB 3″ long has its end A in the ground line. AB makes 30° with the horizontal plane and 40° with the vertical plane. (a) Find the lengths of the plan and elevation. *Ans.* 2.60″, 2.30″. (b) Find the distances of B from the horizontal and vertical planes. *Ans.* 1.5″, 1.93″. (c) Represent the line in plan and elevation.

6. The projectors of the ends of a line are 1.5″ apart. The plan measures 2″ and the line makes an angle of 32° with the vertical plane. Represent the line in plan and elevation. Find the inclination of the line to the horizontal plane. *Ans.* 45°.

177. PROBLEM.—**A line AB 2.5″ long, has its ends in the planes of projection. The line makes 35° with the horizontal plane and 20° with the vertical plane. Draw the plan and elevation of AB. Scale $\frac{1}{4}$.**

Construct Fig. 176 to the given data, making OP = AB = 2.5″, MOP = 35°, NOP = 20°.

Now draw any line ab equal in length to OM. Take this for the plan of AB.

With centre a, radius PN, describe a circle. Through b draw a tangent to the circle, and take this tangent for the ground line xy.

Draw the elevation of AB on this ground line, setting off the height of b′ above xy equal to PM; *or* making the length of the elevation a′b′ equal to ON.

Apply Prob. 174 to the answer, and verify that the length and angles are correct.

178. PROBLEM.—**To find the traces of a given line AB on the planes of projection.**

Note.—The line of intersection of two surfaces is sometimes called the **trace** of one surface on the other. The point of intersection of a line and surface is called the **trace** of the line on the surface. We thus have the **horizontal** and **vertical traces** of lines and planes on the two planes of projection.

First method.—Produce the elevation b′a′ to meet xy in h′. Then h′ must be the *elevation* of the horizontal trace. Draw a projector from h′ to meet the plan ba produced in h. Then h is the *plan* of the horizontal trace.

Produce the plan ab to meet xy in v, and from v draw a projector to meet the elevation produced in v′. Then V is the vertical trace.

Second method. By rabattements.—The traces could evidently be found from the rabattements of the lines as shown. See Prob. 174.

Note.—If the line was situated as in (4), p. 185, the first method would fail to give a result, but the second method would apply. Let the student try this case.

179. PROBLEM.—**A triangle PQR is given by its figured plan. Determine the true shape of the triangle. Scale $\frac{1}{4}$.**

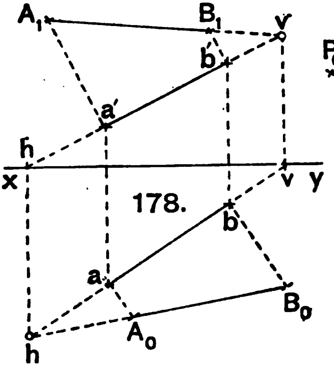
Rabatte the three sides into the horizontal plane about their plans as shown; consult Prob. 173. The lengths of the three sides now being known, the true shape of the triangle PQR can be set out. It is the triangle $P_0Q_0R_0$.

Note.—$P_0Q_0R_0$ may be regarded as the rabattement of the triangle PQR into the horizontal plane about pq and PQ as hinges. The triangle is shown inverted, after rabattement.

180. PROBLEM.—**Find the angle subtended at the origin by a line AB, the co-ordinates of whose ends are (2.5″, 1.5″, 0.5″), and (0.5″, 4″, 2″). No figure.**

Draw the axes OX, OY, and set out the plans of A and B. Then by means of rabattements or right-angled triangles find the lengths of OA, OB, and AB.

Using these lengths, plot the true shape of the triangle OAB, and measure the angle O. *Ans.* 51.6°.



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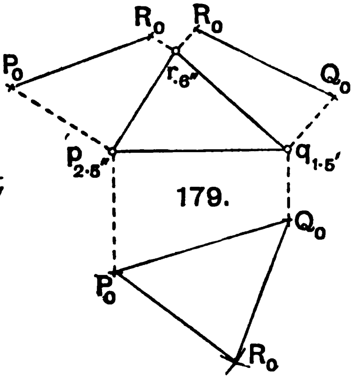


179.

181. Miscellaneous Examples.

1. The rectangular co-ordinates of the ends of a line AB are (2.5″, 1.5″, 0.5″), and (0.5″, 4″, 2″) :—
   - (a) Represent the planes of reference and the line AB in pictorial projection.
   - (b) Draw the three projections of AB, and set up the model. See Fig. (c), p. 183. Take O 3½″ from the two edges of the drawing-paper.
   - (c) Find the length of AB and the angles which AB makes with the planes of XY, YZ, and ZX. *Ans.* 3.53″, 25°, 34.6°, 44.7°.

     Verify numerically, using the tables on p. 24, that $\sin^2 25° + \sin^2 34.6° + \sin^2 44.7° = 1$.
   - (d) Show the traces of AB on the planes of XY and YZ, and measure the co-ordinates of the traces. *Ans.* (3.17″, 0.77″, 0), (0, 4⅝″, 2⅜″).
   - (e) Determine the lengths of OA and OB. *Ans.* 2.96″, 4.50″. Set out the true shape of the triangle OAB and measure the angle AOB. *Ans.* 51.6°.
   - (f) Represent AB by a plan and elevation. Show the two rabattements of AB into the planes of projection, and the two traces of AB, as in Prob. 178. Draw the pictorial view for this case and find the traces in this view.
   - (g) Represent AB by a figured plan. Show the rabattement and the horizontal trace of AB.
2. Find the distance between opposite corners of a building brick

$9'' \times 4\frac{1}{2}'' \times 3''$. *Ans.* $10\frac{1}{2}''$. Find also the angles which the line joining two opposite corners makes with the edges and faces of the prism. *Ans.* 31°, 64.7°, 73.4° and their complements.

3. A line CD is 3″ long; it is inclined at 45° to the horizontal plane and 30° to the vertical plane, and the end C is in both planes of projection.

(a) Determine the height of D and the length of the plan cd, and draw a figured plan of the line. *Ans.* 2.12″, 2.12″.

(b) Determine the distance of D from the vertical plane (*i.e.* of d from xy) in order that the line may make 30° with the vertical plane. Then draw xy through c to satisfy this condition. *Ans.* $3'' \sin 30° = 1.50''$.

(c) Draw the elevation of CD on xy. Verify the result by examining whether $c'd' \div CD = \cos 30° = 0.866$.

***4.** You are given a pictorial projection of the planes of reference and a line AB drawn half size. Copy the figure full size; then

(a) Measure the co-ordinates of A and B. *Ans.* (1.9″, 0.4″, 2.1″), (0.39″, 2.67″, 0.88″).

(b) In this view show the vertical and horizontal traces of AB and measure their co-ordinates. *Ans.* (0, 3.55″, 0.45″), (−0.56″, 3.45″, 0).

(c) Find the true length of AB and the inclination of AB to the plane of XY. *Ans.* 2.95″, 24°.

***5.** The plan OM, ON is shown of one corner of a room. Scale, 1″ to 10 feet. Two small objects P and Q in the room are shown by their figured plans, indexed in feet above the floor.

(a) Find the distance apart of P and Q, and the distance of each point from the corner O of the floor. *Ans.* 28.7′, 21.4′, 27.0′.

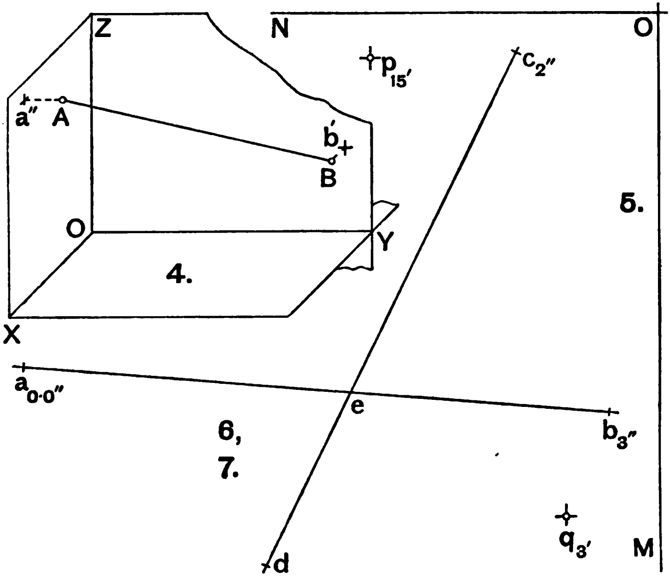
(b) If two strings were stretched from O to P and Q, what would be the angle between them? *Ans.* 71.5°.

(c) Find the angles which the line joining PQ makes with the floor and the walls OM and ON. *Ans.* 25°, 21°, 58°.

***6.** Two lines AB, CD in space intersect at E. Their plans are given, the heights of A, B, and C being indexed. Determine the heights of E and D, and index these heights on the plan. *Ans.* 1.67″, 1.50″.

***7.** Find the angle AEC. *Ans.* 112°. Set out the true shape of the quadrilateral ABCD.

8. A is a point in the vertical plane $1\frac{1}{4}''$ above the horizontal plane; B is a point in the vertical plane $1\frac{3}{4}''$ from the vertical plane. The real distance from A to B is 3″. Draw the plan and elevation of the line AB.



9. A point is 2″ from the vertical plane and 1.75″ from the horizontal plane. Determine a point on the ground line distant 3.25″ from this point.

10. A point 1.5″ from both planes of projection is distant 3.2″ from another point which is 2.3″ from both planes of projection. Obtain the projections of the two points.

CHAPTER XVII

THE PLANE IN SPACE

182. Methods of representing a plane in space.—A plane, supposed to be of indefinite extent, is located in space when the positions of any three points in it, not in one line, are known. In the rectangular system of co-ordinates, the points where the plane cuts the axes, say A, B, C are the three most convenient points to choose for this purpose. The lengths OA, OB, OC are called the *intercepts* of the plane on the axes. The lines AB, BC, CA where the plane cuts the planes of reference are called the *traces* of the plane. These traces serve very well to represent a plane by projection, and any two of the three are sufficient. We therefore see that

(a) The position of a plane in space may be defined by a numerical statement of its intercepts on the axes. As OA = 1.6″, OB = 2.5″, OC = 1.7″.

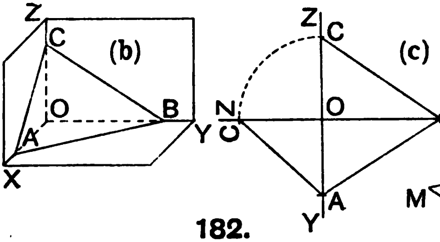
(b) The plane may be shown by a pictorial projection of its traces.

(c) The position of the plane may be exhibited by its three traces on the rabatted planes of reference.

(d) A plane may be represented by its two traces on the two planes of projection. See pp. 197 to 199.

(e) A plane may also be represented by a figured plan or scale of slope. See pp. 200 and 201.

Compare these different ways of representing a plane with the corresponding methods for the line, Art. 170.



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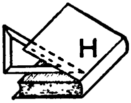
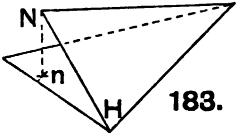




183.

183. The angle between two planes.—How is the angle between two planes, called a **dihedral angle,** measured? The student probably knows quite well.

He understands, for instance, that the inclination of a hillside is measured by that of a *path of steepest ascent*, and knows that a path in any other direction is less steep.

He could adjust his 30° set-square properly between the cover and leaves of a book so as to ensure that the angle between the two surfaces was 30°. See the illustration. He would, quite naturally, place the two edges or the plane of the square perpendicular to the hinge H where the planes meet.

Model.—Let an irregular quadrilateral HKMN made in sheet metal be bent along a diagonal as shown. It may be painted black for the use of chalk, and should be entirely devoid of construction lines. This model being given to a student, he should be asked:

(**a**) To measure the dihedral angle between the two faces.

(**b**) To take dimensions of the model, and to represent it to scale by a pictorial projection, a plan and elevation, or by a figured plan, when standing on one face. And

(**c**) To determine from his drawing the angle between the faces, using a construction of his own invention, comparing his answer with the previous measurement in (**a**).

Let the student take the model, or, failing that, a folded sheet of paper, and from any point O in the intersection HK, draw two lines OL, OL perpendicular to the intersection, one in each plane. He will at once see that the plane LOL is perpendicular to HK; that the lines OL, OL are projections of one another on the two faces; and that the angle LOL measures the dihedral angle between the faces, in whatever position in space the model may be held. We thus have the following three equivalent rules:—

Rules for measuring a dihedral angle.—The angle between two planes is measured:—

1. *By the angle between two lines lying one in each plane, both perpendicular to the intersection.* Or
2. *By the angle between the two lines in which they are cut by a plane perpendicular to the intersection.* Or
3. *By the angle between a line in one plane perpendicular to the intersection, and its projection on the other plane.*

For example, the dihedral angle between the front and side sloping roof surfaces of a house is equal to the angle between two lines chalked on the slates, both perpendicular to and meeting in the common ridge.

Example.—Failing an actual sheet metal model, find the dihedral angle between the faces of the model represented in pictorial projection, Fig. 183. *Ans.* 33°.

184. PROBLEM.—**To find the angles which a given plane makes with the planes of projection. Also to find the angles between the traces. Let the plane be defined by its intercepts OA = 2.5″, OB = 3.3″, OC = 2″. Scale $\frac{1}{4}$.**

(a) The pictorial view is set out to scale.

(b) The three traces are shown on the rabatted planes of reference, conveniently placed for the model.

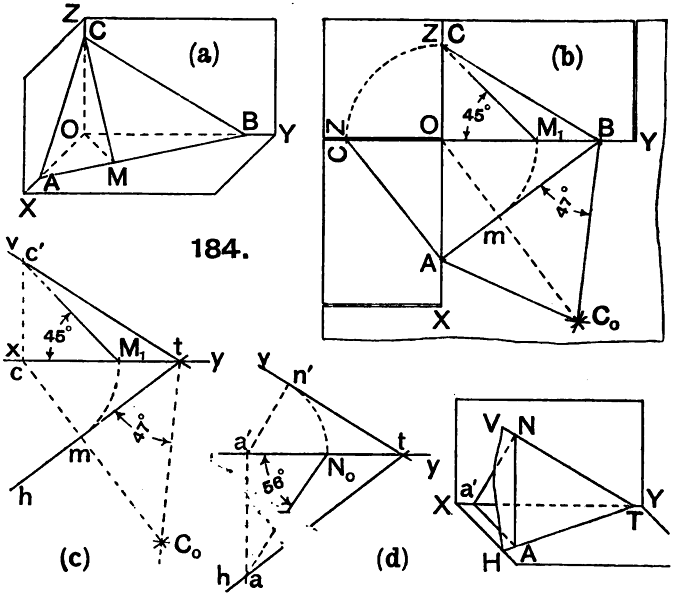
Inclination to horizontal plane.—Let CM, Fig. **(a)**, be a line of *steepest descent* on the plane, and therefore perpendicular to AB. Its plan is OM. We require to find the base angle of the right-angled triangle COM.

The true length of the base is Om, Fig. **(b)**; the height is OC. This triangle is shown at COM_1 rabatted into the plane YZ about OC. Its base angle measures 45°.

Angles between the traces.—These are simply the angles of the triangle ABC. This triangle is shown at ABC_0 rabatted into the horizontal plane about AB. The rabattement may be plotted by making the sides AC_0, BC_0 equal to AC, BC; *or* by setting off the altitude mC_0 equal to the true length of MC; *i.e.* to the hypotenuse M_1C. The angle ABC measures 47°.

Model.—The student should cut his paper so that the triangles ABC_0, OM_1C, as well as the planes YZ, ZX can be turned into position.

(c) **Plane given by two traces.**—The same plane is



184.

shown, but omitting the side plane ZX and the trace AC, The construction for obtaining the two angles is repeated. so that the student may see how it looks under the new conditions. Observe that tC_0 is the rabattement of the vertical trace vt about the horizontal trace ht.

(d) Inclination to the vertical plane.—Let NA be perpendicular to VT; its elevation Na′ is also perpendicular to VT. We require the angle N of the right-angled triangle ANa′.

The true length of a′A is a′a; that of a′N is a′n′. The triangle is shown at $aa'N_0$ rabatted into the horizontal plane about aa′. The angle at N measures 56°.

The inclination to the plane ZX we leave to be found by the student. The answer is 64.5°.

185. **Planes in various positions.**—In the figure eight planes are represented by their traces, all situated differently as regards the two planes of projection. Pictorial views are shown to assist in rendering the positions clear.

1. This plane is perpendicular to the ground line.
2. The plane is at right angles to the vertical plane of projection, and inclined to the horizontal plane. It is often called an **inclined plane.** The horizontal trace is perpendicular to xy.
3. This is a **vertical plane,** inclined to the vertical plane of projection. Its vertical trace is perpendicular to xy.

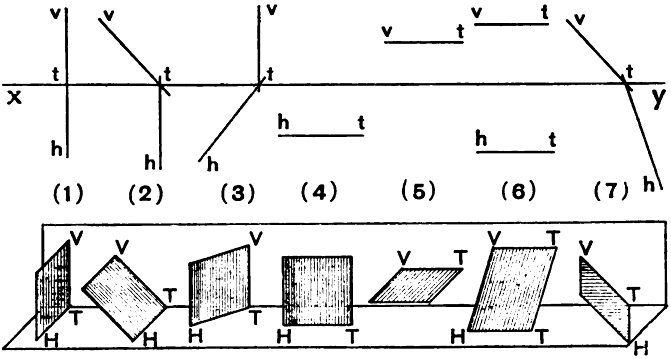
   In the above three planes the angle between the traces is 90°.
4. The plane is parallel to the vertical plane of projection. The horizontal trace is parallel to xy. There is no vertical trace.
5. The plane is horizontal and the vertical trace is parallel to xy. The horizontal trace may be considered at an infinite distance away and parallel to xy.
6. The plane is parallel to xy and inclined to both planes of projection. Both traces are parallel to the ground line. A plane inclined to both planes of projection is often called an **oblique plane.**
7. This is an oblique plane in which the angle between the traces is obtuse. In the oblique plane of 186 below, the angle is acute. Neither trace can be perpendicular to xy.
8. An oblique plane containing xy. It is represented by an edge view on the plane of ZX.

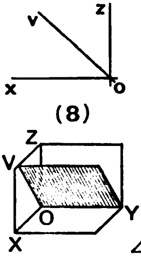
186. PROBLEM.—**A plane VTH is given, and also the plan of a point A lying in it. Show two lines lying in the plane and containing A, one SL being a line of steepest slope, the other MN being horizontal.**

Also determine the elevation of A, and the rabattement of A into the horizontal plane about the horizontal trace HT.

Through a draw the line sl perpendicular to ht. This is the plan of the line of steepest slope, its ends S and L being in the planes of projection, and in the traces of the given plane. Project the elevation s′l′.

Next, through a draw the line mn parallel to the horizontal trace th. This is the plan of the required horizontal line, its end M being in the vertical plane of projection, and in the vertical trace VT. Project from m to m′, and draw the elevation m′n′ parallel to xy.





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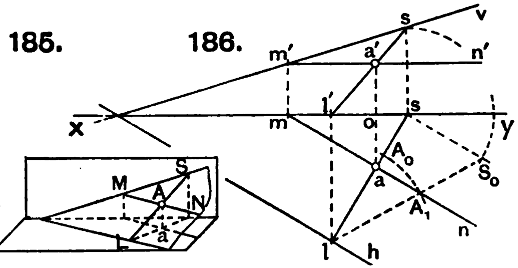


186.

Elevation of A. Project from a on to either s′l′ or m′n′.

Rabattement of A about HT. First determine the true distance *up the plane* from L to A. This is the hypotenuse of the right-angled triangle LaA, base la, height aA =oa′. This triangle is shown at laA_1 rabatted about its base. Now in la produced mark off lA_0 equal to the hypotenuse lA_1. Then A_0 is the required rabattement of A. Observe that the base angle alA_1 gives the **inclination of the plane.**

Note.—All the parts of this problem are most important, and should be thoroughly mastered by the student. To assist him we have drawn a pictorial view, half size, in which the lines SL, MN and their plans are shown. Let the student devise a model in further illustration. See also Art. 189.

187. How a plane is represented by one projection.—The student will easily think out several ways of doing this. One feasible method is to draw the horizontal trace and write down the inclination in degrees. Another is to draw the plan of any line of steepest slope, perpendicular to the horizontal trace, and to index any two points in it.

The method usually adopted is an extension of the latter, a number of points on a line of steepest slope being indexed, so as to form a scale called a **scale of slope**, from which the height of any point in the plane can be very readily ascertained. The following problem will illustrate the method:—

188. PROBLEM.—**The given line hh is the horizontal trace of a plane, which has an upward inclination towards the right of 35°. Represent the plane by a scale of slope, indexed in millimetres.**

Take any ground line xy perpendicular to hh cutting the latter in t. Draw tv at 35° to xy. Then tv is the vertical trace or edge elevation of the given plane on a vertical plane of projection through xy.

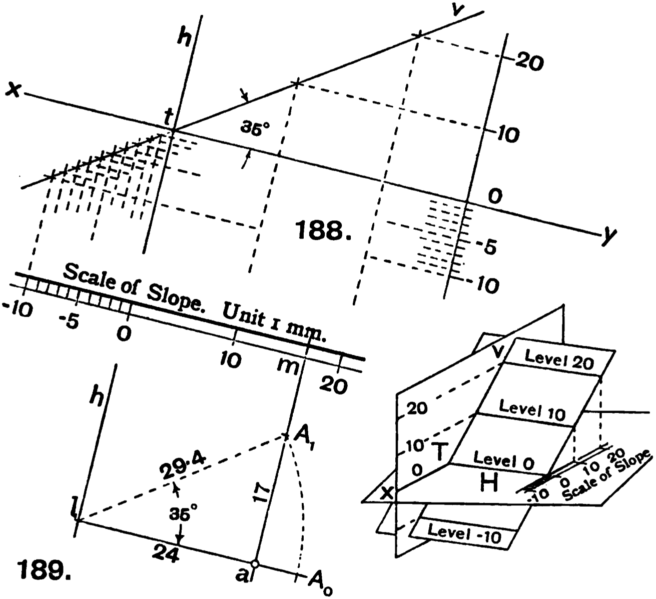
Along any line at right angles to xy mark off a vertical scale of heights, in millimetres, the zero point being on xy, as shown. From the points of division draw lines parallel to xy cutting tv. These lines may be regarded as the vertical traces or edge elevations of horizontal planes at the several levels; compare (5) Art. 185.

Now draw any two lines perpendicular to HH, about $\frac{1}{16}''$ apart as if for a scale. And from the points on tv project a scale and figure it as shown. This scale is the required scale of slope, and represents the given plane.

The projectors last drawn are the plans of horizontal lines in the plane at the several levels. The scale itself is the plan of short portions of these lines of level, and of two adjacent lines of steepest slope. See the pictorial view, which is drawn to scale, half size.

189. Use of the scale of slope.—Suppose only the scale of slope and the plan a of a point in the plane, were given.

(a) Through the zero point draw the line hh perpen-



188.

189.

dicular to the scale. Then hh is the **horizontal trace** of the plane.

(b) Through a draw am perpendicular, and al parallel to the scale. Let these represent lines in the plane. Then AM is a **line of level**, and AL is a line of **steepest descent.**

(c) Measure la or om on the scale of slope; the result, 17 mm. gives the **vertical height** of A above L, or the index of a.

(d) Measure la on the linear scale, the result 24 mm. is the **horizontal distance** between L and A.

(e) Set out the right-angled triangle laA_1, height aA_1 = 17 mm.; the hypotenuse lA_1, measuring 29.4 mm., is the **distance up the plane,** from L to A. Measure its base angle alA_1; the result, 35°, gives the **inclination** of the plane.

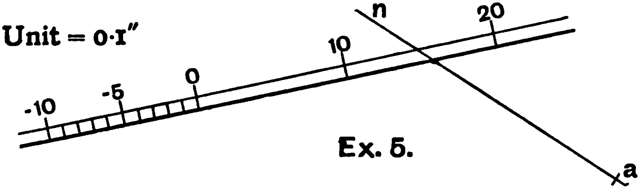
(f) On la produced, mark off lA_0 equal lA_1. Then A_0 is the **rabattement of** A about the horizontal trace of the plane.

Compare this Article with Prob. 186.

190. Examples on the plane.—Before proceeding farther, the student must make sure of the ground already covered by working some examples.

1. The intercepts of a plane on the axes of X, Y, and Z are OA = 3.3″, OB = 2.5″, OC = 2″.
   (a) Draw the pictorial view showing the three traces.
   (b) Draw the traces on the rabatted planes of reference, and set up the model. See Fig. 184 (b); take O 2½″ from the two edges of the paper.
   (c) Draw the rabattement of the two vertical traces into the plane XY about the horizontal trace AB. Measure the lengths of the traces and the angles between them. *Ans.* AB = 4.14″, BC = 3.86″, CA = 3.20″. ABC = 47°, BCA = 71°, CAB = 62°.
   (d) Find the angles which the plane ABC makes with the planes XY, YZ, ZX. *Ans.* 45°, 56°, 64.5°. Verify by calculation, using the tables p. 24, that $\cos^2 45° + \cos^2 56° + \cos^2 64.5° = 1$.
   (e) On the model show the triangle ABC in position, also the right-angled triangle COM, Fig. 184, the base angle of which gives the inclination of the plane.
   (f) Draw a pictorial view showing only the two planes of projection, and the horizontal and vertical traces of the plane.
2. The vertical trace of a plane makes 30° with xy and the horizontal trace 45°. Find the inclinations of the plane to the horizontal and vertical planes of projection, and the angle between the traces. *Ans.* 39.3°, 63.5°, 52°.
3. A plane is inclined at 60° to the horizontal, and its horizontal trace makes 45° with xy. Draw the vertical trace. Find the inclination of the plane to the vertical plane and the angle between its traces. *Ans.* 52.3°, 63.3°.
4. The real angle between the traces of a plane is 60°. The horizontal trace makes 45° with xy. Draw the vertical trace, and measure the angle it makes with xy; also determine the angles which the plane makes with the horizontal and vertical planes of projection. *Ans.* 45°, 54.7°, 54.7°.

*5. The scale of slope of a plane is given, heights being figured in tenths of an inch. The plan of a line AN lying in the plane is also given.



Ex. 5.

- **(a)** Draw the horizontal trace of the plane.
- **(b)** Determine the height of A and index its plan. *Ans.* 2.59″; index 25.9 (units).
- **(c)** Measure the inclination of the plane. *Ans.* 51.4°.
- **(d)** Draw the plan of the shortest path AL down the plane from A to the horizontal trace. Measure the length of this path, and its inclination. *Ans.* 3.31″, 51.4°.
- **(e)** Find the length and inclination of a path which goes in the direction an in plan and terminates in the horizontal plane. *Ans.* 3.91″, 41.6°.
- **(f)** Find the real angles between the paths AL, AN and a horizontal path AM. *Ans.* LAN = 31.9°, NAM = 58.1° or 121.9°, LAM = 90°.

6. The lines of steepest ascent up the plane face of a hill go eastwards. The slope is 1 vertical in 3 horizontal.
   - **(a)** Represent the surface of the ground by a scale of slope, the linear scale of the plan being $\frac{1}{4}$ inch to 100 feet. What is the inclination of the hill in degrees? *Ans.* $\tan^{-1}\frac{1}{3} = 18.5°$.
   - **(b)** What is the slope and inclination of a path which goes N.E.? *Ans.* 1 in $3\sqrt{2}$; $\tan^{-1} 0.236$ or 13.2°.
   - **(c)** What is the inclination of a northward path? *Ans.* 0°.
   - **(d)** What is the direction of an upward path which rises 1 in 7? *Ans.* $\cos^{-1}\frac{3}{7}$ or 64.6° N. or S. of E.
   - **(e)** What would be the length of this path for a vertical rise of 160 feet? *Ans.* $160\sqrt{1+7^2}$ feet = 1131 feet.

Note.—The student should draw the plans of these several paths, and obtain the answers graphically.

7. The plan of an equilateral triangle is an isosceles right-angled triangle. Determine the inclinations of its sides and plane. *Ans.* 0°, 45°, 45°; and 54.7°.
8. Draw to scale the plan of your 30°-60° set-square when resting on its long edge, with its plane inclined so that the plan of the right angle measures 120°. Find the inclinations of its plane and edges.

191. Problem.—**A plane is given by its traces; the plan of a point A in the plane is also given. Find the elevation of A. Determine a line AB which lies in the plane and is inclined to the ground at 37°. Scale, $\frac{1}{2}$.**

Determine a′ by means of the horizontal line am, m′a′, as in Prob. 186.

Draw $a'B_1'$ at 37° to xy, and draw its plan aB_1 as shown. This represents a line parallel to the vertical plane, inclined at 37° to the horizontal plane, with its upper end in the oblique plane and its lower end on the ground.

Now let the end A of the line remain fixed, while the lower end is turned into the horizontal trace, bringing B_1 to b, and B_1' to b′. The line now lies altogether in the plane, and it is still inclined at 37°.

192. Problem.—**A plane is given and also the plan ab of a line AB lying in it. Find the elevation, true length, and inclination of AB. Scale, $\frac{1}{2}$.**

Project from b to b′. Obtain a′ as before by the aid of a horizontal AM in the plane. Join a′b′.

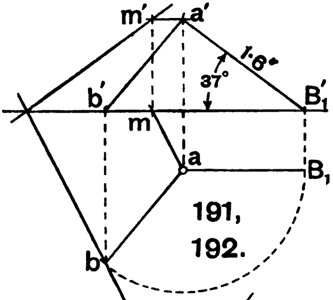
Next let AB be turned about the vertical axis Aa until it is parallel to the vertical plane. This brings b to B_1 and b′ to B_1'. The true length and inclination are now exhibited in elevation; $a'B_1'$ measures 1.6″, and its inclination measures 37°.

Note.—When the *inclination* of a line is referred to without qualification, inclination to the *horizontal plane* must always be understood.

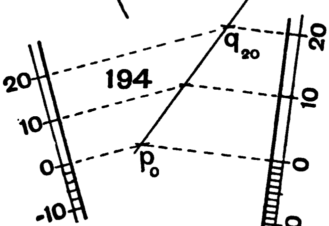
193. Problem.—**Determine the intersection of a line and plane given by their figured plans. Linear scale, $\frac{1}{4}$. Unit for heights, 0·1″.**

Take a ground line xy parallel to the scale of slope. Project the elevation of the line. Also, by projection from the scale of slope, determine the vertical trace vt or *edge elevation* of the plane.

The point s′ where these two elevations intersect, is the elevation of the required point. Project from s′ to s. Measure the height of s′, viz. 1.5″ or 15 units, and index s.

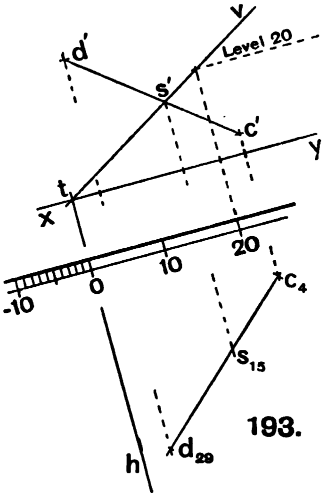


191,
192.



194

Scale ½
Unit 0·1″



193.

Linear Scale ¼
Unit for Heights 0·1″

194. PROBLEM.—**Find the intersection of two planes given by their scales of slope. Scale, $\frac{1}{2}$. Unit for heights, 0·1″.**

Draw the horizontal traces of the planes, *i.e.* lines through the zero points, perpendicular to the scales. These intersect in p_0, and give one point in the common intersection.

Draw the plans of another pair of horizontal lines in the planes, both at the same height, say at the level 20. These intersect in q_{20}, and determine a second point in the intersection.

Join these two points. The intersection PQ is thus defined by its figured plan.

Example.—Find the inclination of PQ. *Ans.* 53°.

195. PROBLEM.—**Find the intersection of the two planes given by their traces. Measure its inclination and the length of the portion terminated by the planes of projection. Scale $\frac{1}{2}$.**

Let the horizontal traces intersect in n and the vertical traces in m′. Project from m′ to m and from n to n′ as shown. Join m′n′, mn. These lines are the projections of the required portion M N of the line of intersection. See the pictorial view.

To find the length and inclination of M N, we have rabatted the line into the horizontal plane about its plan. The length measures 2.36″ and the inclination 26°.

The lower figure shows a second case in which an inclined plane and vertical plane intersect. The data and results are figured on the diagram.

196. PROBLEM.—**To find the angle between two planes which are given by their traces.**

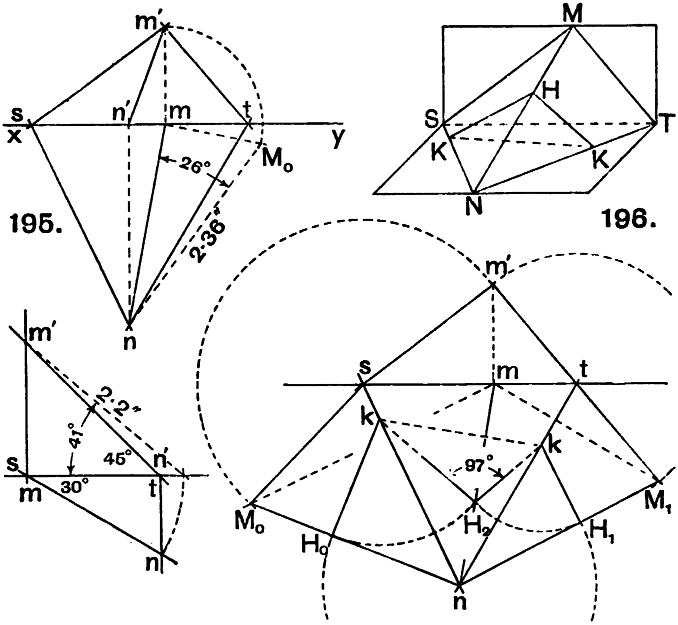
First refer to the pictorial view. Let H K, H K be lines in the two planes, both perpendicular to M N. The angle between them measures the dihedral angle between the planes. Consult Art. 183. To find this angle we shall draw the true shape of the triangle H K K, and to find the lengths of its sides we shall first turn the two triangles M S N, M T N into the plane of the paper.

Now refer to the construction below. Find the plan mn of the intersection as in Prob. 195. Rabatt the two planes, *i.e.* the two triangles M S N, M T N about their bases, obtaining nM_0s and nM_1t. See Prob. 184.

Mark off from n any convenient *equal* lengths nH_0, nH_1 as shown. Draw H_0k, H_1k perpendicular to nM_0, nM_1. On the base kk draw the triangle kH_2k, with its sides equal to kH_0, kH_1.

This triangle is the rabattement of K H K about K K. Its vertical angle measures 97°.

Model.—Let the student cut out the figure $sM_0nM_1tm's$ from his drawing-paper. Then cut the paper through along kH_2k, and indent it along sn, nt, ts, kk. The four triangles



195.

196.

may now be turned into their true relative positions in space.

197. Miscellaneous Examples.

1. The length of the plan of a line measures 1″. The difference in the figured heights of its ends is 1″. What is the inclination of the line? *Ans.* 45°.
2. In the scale of slope of a plane, two adjacent main divisions are marked 0 and 10, and the unit for heights is 0.1″. What is the inclination of the plane, if the main division measures (a) 0.70″, (b) 1.0″, and (c) 1.4″? *Ans.* 55°, 45°, 35.5°.
3. A length on a scale of slope representing the plane face of a hill measures 300 feet on the linear scale of the plan, and the difference of height indexed is 100 feet. What is the slope and inclination of the hill? *Ans.* 1 in 3, 18.5°.

***4.** The point D lies in the plane ABC. Find the distance of D from O. Scale half size. Axes and scale ratios as before.

Show in the pictorial view a perpendicular OP on the plane ABC. Find the length of this perpendicular, and the angles it makes with the axes.

***5.** A 45° set-square is shown in plan at abc, the edge BC resting on the ground. Scale $\frac{1}{4}$. A post-card rests against the edge AB. Find the lengths and inclinations of the three edges of the squares. Find also the dimensions and inclination of the post-card. Draw an elevation on xy. Scale $\frac{1}{2}$ (*i.e.* double the size of the diagram).

***6.** The plan of the roof of a house is given; scale 1 cm. to 10 feet. At the front and back the inclination of the roof surface is 27°, find that at the sides. Find also the inclination of the four sloping ridges. What is the total area of the surface in square feet? What is the dihedral angle between the front and side surfaces of the roof? Adopt the scale of 1″ to 10′.

***7.** A piece of sheet brass cut to the given dimensions is bent along AC so that the angle between its faces is 55°. Draw its plan when resting on the face ABC. Scale $\frac{1}{2}$. What is the height of D and the distance of D from B? What is the inclination of AD and the angle between AB, AD?

Draw the plan when the model is turned over so as to rest on the edges AB, AD.

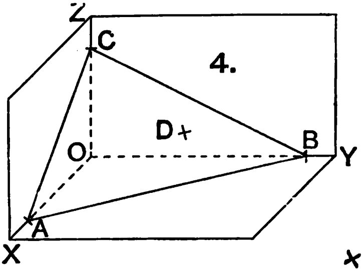
***8.** The plans are shown of two lines, MN, MN, which lie in the given plane. Scale $\frac{1}{2}$. Draw, full size, the elevations of the lines. Obtain the rabattement of the lines about NN, and measure the true angle NMN.

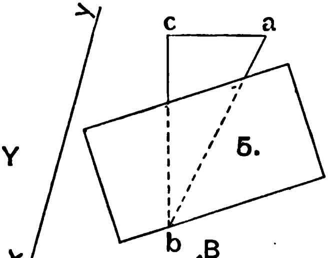
***9.** The figured plans of two lines, SR, ST are given. Scale $\frac{1}{2}$. Unit for heights 0.1″. Copy these full size. Obtain the traces of the lines, and the trace of the plane RST. Find the rabattement of the plane and lines. Draw the plan of the bisector of the angle RST. Show the trace of this bisector.

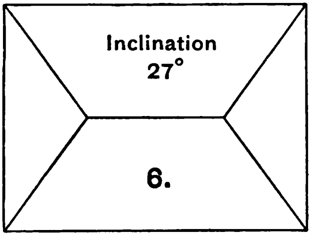
***10.** The scales of slope of two planes, and the figured plan of a line LL are given. Scale $\frac{1}{2}$. Unit 0.1″. Copy the figure full size, and determine the figured plans of the two points where LL cuts the planes, and give the length of this intercept.

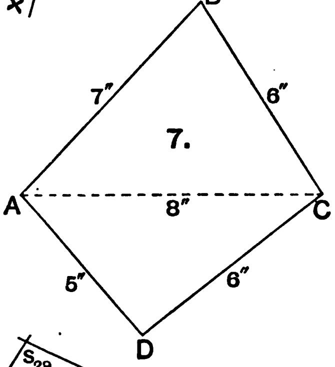
Copy the figure again, and find the intersection of the planes. Also determine the dihedral angle between them.

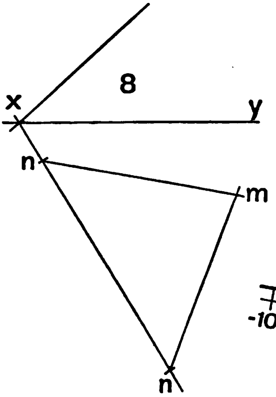
11. A person P on the top of a tower 60 feet high, which rises from a horizontal plane, observes the angles of depression of two objects A and B on the plane to be 20° and 30°, the direction of A and B from the tower being west and south respectively. Represent A, B and P by a figured plan, indexed in feet, the linear scale being 1 inch to 100 feet. Measure (a) the dis-

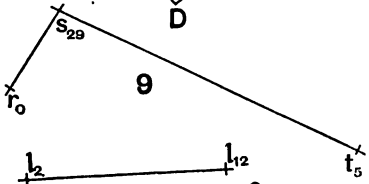


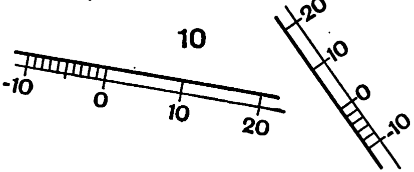












tances of A and B from the tower; (**b**) the distances of A and B from the person; (**c**) the distance apart of A and B; and (**d**) the direction of B from A. *Ans.* (**a**) 165′, 104′; (**b**) 175′, 120′; (**c**) 195′; (**d**) E. 32.2° S.

CHAPTER XVIII

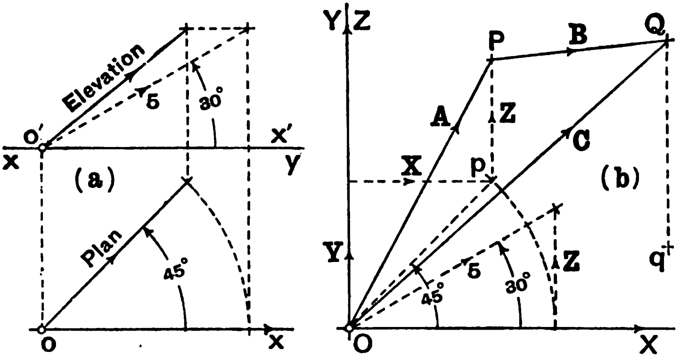
MISCELLANEOUS PROBLEMS AND EXAMPLES IN PLANE AND SOLID GEOMETRY

198. Vectors in space.—Vectors which have any directions in space are added in just the same manner as are vectors parallel to a plane. They are placed end to end circuitally in order to obtain the non-circuital closing vector. The polygon is no longer a plane figure, and in order to be able to represent it by drawing, we must have recourse to the methods of projection.

In specifying a plane vector (unlocalised), the reader will remember that we use the notation A_α, say 5_{45°, signifying thereby 5 units in the direction N.E., the plane being taken as horizontal, and it being understood that angles are always measured anti-clockwise from the east.

If now the vector act in the N.E. vertical plane, but at an angle of elevation above the horizontal of β, say 30°, this additional information must be added to the specification. This may be done by writing the vector as A_α^β. Thus $5_{45^\circ}^{30^\circ}$ signifies the vector:—Magnitude, 5 units; horizontal direction, 45° northwards of east; vertical direction, 30° above the horizontal. This vector is represented in plan and elevation at (a), scale $\frac{1}{4}$, the plotting of which is evident from the figure.

If the method of pictorial projection be adopted, the axis of OX would be the initial or eastward line, anti-clockwise angles being measured towards OY. It might be convenient to alter the system of axes and scales



hitherto adopted, and, regardless of apparent distortion, represent OY and OZ as coincident, and both perpendicular to OX, as shown at (b), the scales being taken the same for all three axes. Plans and front elevations would then appear of their actual shape, and side elevations would all be straight lines coinciding with OYZ, which is now an edge view of the plane YZ.

The vector $\mathbf{A} = A_\alpha^\beta = 5_{45°}^{30°}$ is thus represented in (b), scale $\frac{1}{4}$. Its components parallel to the axes, which we have denoted by **X**, **Y**, **Z**, are shown. The same scale of $\frac{1}{4}$ is to be used for all these; they measure 3.06, 3.06, and 2.5 units respectively.

A second vector **B** is shown added on to **A**. Its components, as well as those of the resultant $\mathbf{A}+\mathbf{B}$ or **C**, can be directly scaled from the figure, and the magnitudes and angular elements α and β ascertained by simple constructions of right-angled triangles. See Ex. 2.

Examples.—1. Find the rectangular components of the vector $8_{30°}^{45°}$. *Ans.* 4.90, 2.83, 5.66.

2. Obtain from the diagram the elements of the vectors **B** and **C**. Scale $\frac{1}{4}$. *Ans.* $\mathbf{B}=4.29_{20.6°}^{24.6°}$; $\mathbf{C}=8.14_{14°}^{32°}$.

199. PROBLEM.—**Three views are shown of a portion of a bolt with a hooked head. Scale, $\frac{1}{2}$. Draw a sectional elevation on SS.**

Take x′y′ parallel to SS. Project from the points in plan. Mark off heights above x′y′ equal to the corresponding heights above xy.

To project the circular arc we have taken four points in it and found their elevations. A fair curve is then drawn through these points.

The arrow shows the direction of view for the sectional elevation. The portion of the bolt in front of SS is supposed to be removed, as is usual in sectional projections.

200. PROBLEM.—**Two prisms are grouped as shown. Scale, $\frac{1}{10}$. Draw a new elevation on x′y′.**

The method of working is sufficiently evident from the figure. Compare this method of projecting the new elevation with the method of obtaining (**d**) from (**c**), p. 175.

Note carefully that the elevation illustrates the following **important principle**:—

Theorem.—*In parallel projection parallel lines project into parallel lines, all to the same scale.*

This is the principle on which our pictorial projection is based. The projectors are parallel to one another, but inclined to the plane of projection, so that the pictorial views are not orthogonal projections.

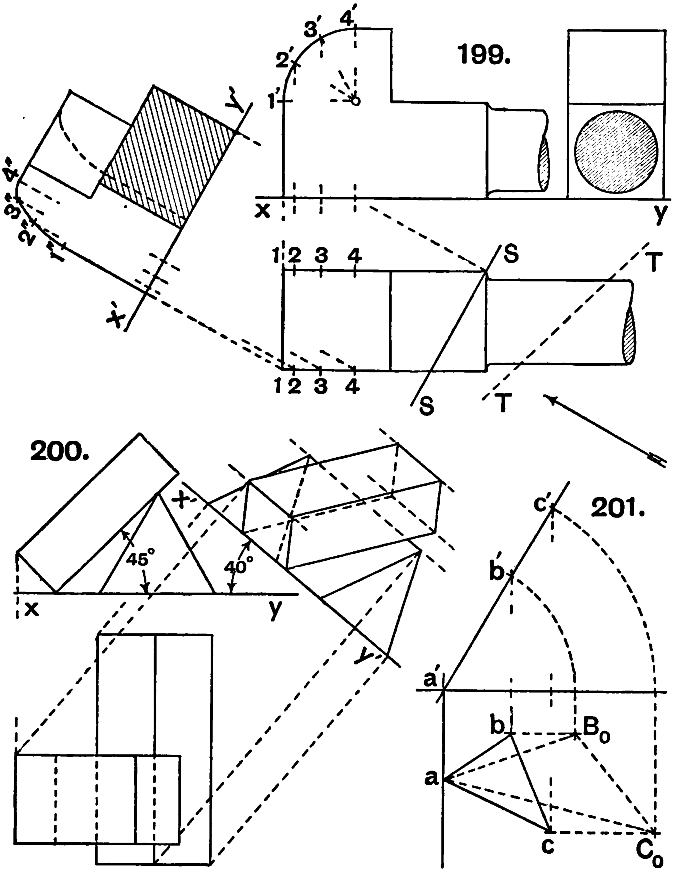
201. PROBLEM.—**The triangle whose plan abc is given lies in the given inclined plane. Obtain the elevation and the true shape of the triangle. Scale, $\frac{1}{2}$.**

The plane is at right angles to the vertical plane of projection, therefore its vertical trace is an **edge elevation** or **profile view** of the plane.

So, to find the elevation, project from a, b, c to the vertical trace, obtaining a′, b′, c′.

For the true shape, rabatte the plane as shown, carrying the triangle to the position aB_0C_0.

Examples.—1. Work Prob. 199, full size, but taking TT for the section plane instead of SS.



2. Work Prob. 200, to scale of ¼, but alter the angle between the ground lines from 40° to 60°.

3. Copy the data of Prob. 201 full size, then work the problem as shown. In addition, find the true shape of ABC by rabatting the plane about its vertical trace into the vertical plane of projection.

202. PROBLEM.—**A triangular prism is given, standing upright. Scale $\frac{1}{2}$. Determine its section by the given oblique plane VTH. Find the true shape of the section.**

The device adopted is to draw a new elevation such as will give us an **edge view** of the plane. The direction of view is that shown by the arrow, parallel to the horizontal trace ht, and the new ground line x′y′ must be drawn at right angles to this.

Project the prism on x′y′. To find the edge elevation of the plane, it will be sufficient to project two points in the plane. Select two convenient points, say those whose plans are r and s. The former is in x′y′ and projects into itself. For the latter, the elevation of S on xy is s′; its elevation on x′y′ is s″, obtained by marking off ss″ equal to ss′. Now draw the edge elevation r s″.

The plane cuts the vertical edges of the prism in points whose elevations on x′y′ are a″, b″, c″. Transfer the heights of these points to the elevation on xy, and thus obtain a′b′c′. The plan of the section is abc itself.

For the true shape, rabatte the plane about rh, making use for this purpose of the elevation on x′y′, and proceeding as in Prob. 201.

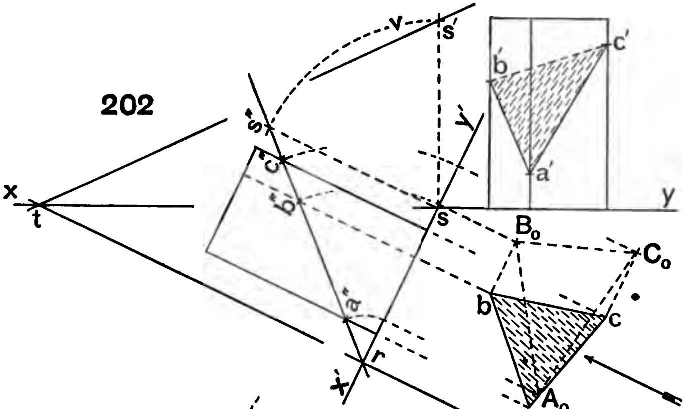
203. PROBLEM.—**A plane figure is given in plan, three of its corners A, B, C being indexed. Scale $\frac{1}{2}$, unit for heights 0.1″. Determine a line of level in its plane. Find the true shape of the figure, the inclination of its plane, and the heights of the remaining corners.**

The heights of B and C as indexed are 2 and 13 units. Let us find a point H in BC, the height of which is 9 units, equal to that of A.

Take a convenient ground line, say xy parallel to bc. Project b′, c′. Draw a line at level 9, cutting b′c′ in h′. Project from h′ to h. Join ah. Then AH is a line of level as required. Lines of greatest slope are perpendicular to AH.

Next take x′y′ *perpendicular* to ah. Project a″, b″, c″. These points will all lie in a straight line, which is the *edge*

202



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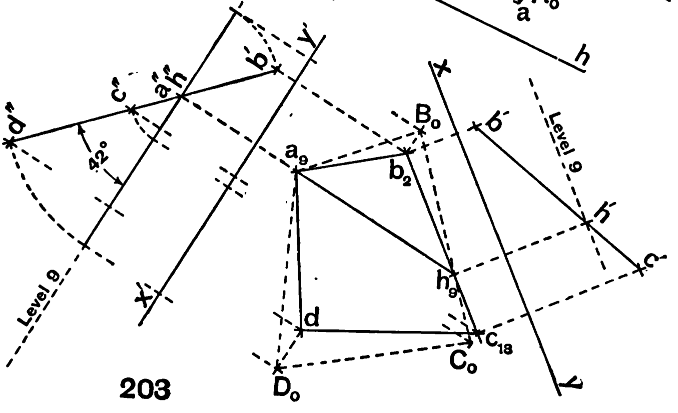


elevation of the plane of the figure. Its inclination measures 42°. Produce the elevation and project from d to d″. The height of D measures 2.2″.

For the true shape turn the figure into the horizontal position about any horizontal line, say about A H as shown. The true shape is $aB_0C_0D_0$.

204. Use of the plane table in surveying. The three point problem.—Let a drawing-board with a truly plane unwarped surface be set exactly horizontal by means of a spirit level. Suppose a straight edge to be pinned to the board at a point exactly in the line of the working edge, so that the straight edge can be turned round this point as a pivot. Suppose the straight edge to be provided with sights, like a gun; but, as the straight edge cannot be elevated, these sights must be vertical lines, and usually consist of a narrow slit and a fine wire.

Having set up this apparatus or **plane table** the straight edge can be brought exactly into line with any visible object, far or near, by the simple operation of sighting or taking aim; and the direction of the object from the pivot can be recorded on the drawing-paper by simply drawing a line along the straight edge.

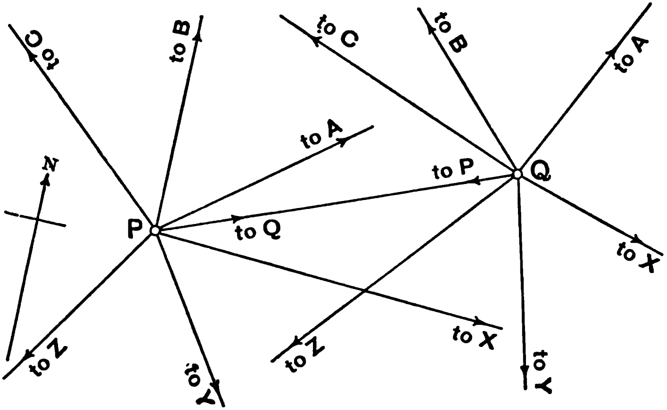
In the figure, let P be the pivot or point about which the straight edge turns, and let the directions of a number of distant objects Q, A, B, C, X, Y, Z, be as drawn.

Now let the same or a second plane table be erected at Q, and the directions of the above objects, *as seen from* Q, be again drawn. The student will see at once that the relative positions of all the points are now located by the intersections of corresponding pairs of lines.

The distance and direction between P and Q may have been directly measured. Say $PQ = 48{\cdot}6_{22^\circ}$ yards, corresponding to the dimensions of the diagram, which is plotted to a scale of 1 cm. to 10 yards. The positions of all the other points can then be measured to scale.

Or the plane table may have been set up in any two convenient but unknown positions, and the relative positions of three of the points in the survey, as A, B, C, may have been directly measured, say $AB = 62_{161.5^\circ}$ yards, $BC = 68.3_{182.3^\circ}$ yards. A, B, C may be rods or poles specially erected for the purpose, or any three suitable objects.

The location of P, or Q on the drawing is then known as the **three point problem.** The practical solution is by



the use of tracing-paper. Observe that by use of the plane table the directions of A, B, C, from P, *i.e.* the angles subtended by the sides of the triangle ABC are known.

Having plotted A, B, C to scale from the measured data, the directions PA, PB, PC are drawn on tracing-paper, this tracing is superposed and adjusted until the lines lie over the points. The point P is then pricked off. A similar process locates Q. Then the remaining points X, Y, Z are known. Note that the construction would fail by being indeterminate, if the points A, B, C, P, or A, B, C, Q, happened to lie all four on the same circle.

Example.—In a rectangular room ABCD the length AB=33′, breadth BC=21′. Draw a plan to the scale of $\frac{1}{8}''$ to 1′. Find a point O on the plan such than AOB=95° BOC=56°. Measure the angle COD. *Ans.* 128°.

APPENDIX I

PRACTICAL PLANE AND SOLID GEOMETRY

SCIENCE SUBJECT I., BOARD OF EDUCATION, SOUTH KENSINGTON

Copy of Syllabus as revised in 1902

Also Copies of the Evening Elementary Examination Papers for 1903 *and* 1904

SYLLABUS

THIS subject comprises the graphical representation of position and form and the graphical solution of problems.

It is intended that it shall include and be confined to those principles underlying the Engineering and Physical groups of subjects which require mechanical drawing with the aid of instruments for their application.

Much lecturing is to be avoided. Students ought to be educated mainly through their own work. A large part of this work should be *quantitative*, and it is important that from the beginning, and throughout the entire course, careful draughtsmanship, and the use of properly adjusted instruments, be insisted on. All constructions should be drawn with hard and finely sharpened pencils, and left without being inked in. Inaccurate work, or work done with soft or blunt pencils, will receive very little credit in the examination, and may even be cancelled.

In the Elementary Stage the main object of the instruction will be to familiarise the student with the fundamental properties of geometrical figures and their applications, and at the same time to train him in the best use of his instruments, and in the habit of measuring his results. This object will be attained principally by the experimental verification and illustration of geometrical truths. It is not intended that the student shall follow Euclid's sequence, nor his reasoning, nor be restricted in the use of instruments by the requirements of such a system of deductive reasoning. Any method which commends itself in enforcing a geometrical truth should be employed. Thus graphical constructions may be supplemented by arithmetic when comparing quantitative results. Tracing-paper will be found useful when applying the method of superposition, or plotting a locus, finding the length of a curve, etc. Squared tracing-paper will sometimes prove helpful. Models should be freely used, especially in solid geometry.

Some suggestions for experiments and exercises are given later in small type. These, however, are to be regarded merely as suggestions; every teacher should develop and illustrate the subject in the way best suited to the particular conditions in which he is placed.

Deductive reasoning and generalisations may be introduced from time to time as the student becomes fitted for them, and when any portions of the subject specially require them.

In the Advanced Stage and in Honours the subject will be developed in its applications to the several branches of the groups of subjects previously mentioned, and will thus tend to become technical. A sufficient choice of questions will be provided, which, including those of a general character, will enable a student who is devoting himself more particularly to any one branch to take the examination. It should be understood that it is better to confine attention to a portion of the subject, and study this thoroughly, than to range over the whole ground in a perfunctory manner.

Compulsory questions may be set at the examinations.

Students may bring tracing-paper to the examination. A copy of the examination tables, p. 78 of the directory, will be supplied to each candidate at the examination.[1]

Stage 1

PLANE GEOMETRY

The drawing pencil and drawing instruments.

The straight-edge, drawing-board, and squares; tests of accuracy and the correction of errors.

The construction of simple geometrical figures designed to test the accuracy of the squares, and to illustrate their use in drawing parallel and perpendicular lines, and lines including angles of 15°, 30°, 45°, 60° and 75°.

Describe a circle, and by use of the tee-square and set-square draw a circumscribing polygon, such as a square or regular hexagon. Test whether the sides or diagonals are of equal lengths, and whether the directions of the diagonals agree with the edges of the squares.

The use of scales, especially scales decimally subdivided in which the lengths of the main divisions are respectively one inch, half inch, quarter inch, one-eighth inch, and one centimetre long. This set of scales and a protractor must be brought to the examination; also inch scales subdivided into eighths and twelfths. With what accuracy may a straight line be drawn and measured? Reasoning on the philosophy of representing things to scale is quite out of place at this stage, and is hurtful to the average student.

Students may experiment for themselves in order to get some idea of the degree of accuracy to be expected in graphical work. They can measure lines and angles, collecting and comparing results. Suitable maps may be measured to illustrate the use of scales; tracing-paper and a pricker may be used with advantage in measuring the length of a road or river.

The construction and measurement of angles from 0°

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[1] See pp. 233-239.

to 360° in degrees and radians, by the use of a protractor, and a scale or table of chords. Limits of error in setting off and measuring angles.

Compare the magnitude of an angle as measured by a protractor and by a table of chords, noting discrepancies. Plot an angle by one method and measure it by the other. By using tracing-paper and a pricker, an arc is readily measured or a given length of arc set off. In this way construct an angle of one radian, making arc = radius, and measure the number of degrees it contains. Also measure the radians in a right angle, or in any angle, and verify by the tables.

The division of lines into equal parts and into parts in given proportions, and other illustrations of the propositions of the sixth book of Euclid.

Let students try and invent for themselves expeditious ways of effecting these important constructions, such as by employing ruled tracing-paper or by bringing their scales into requisition.

The drawing of parallel and perpendicular lines by the aid of compasses. The bisector of an angle and the perpendicular bisector of a line.

Verification of the more important properties of right-angled triangles. Complementary and supplementary angles.

The construction of any right-angled triangle having given (*a*) two sides, (*b*) the hypotenuse and one side, (*c*) the hypotenuse and an adjacent angle, (*d*) one side and the adjacent acute angle, (*e*) one side and the opposite angle.

If possible, some simple apparatus or arrangement should be provided by means of which a student can find the actual height or distance of an inaccessible object from his own observations, and the plotting or solving of right-angled triangles.

The meanings of the sine, cosine, and tangent of an angle. Determination of these ratios for any angle by construction, measurement, and arithmetic. Setting out angles from tables of sines, cosines, and tangents. Solving right-angled triangles by calculation, using the tables of sines, cosines, and tangents.

The fundamental properties of triangles verified and illustrated by drawing. The construction of any triangle having given three independent elements. Equilateral and isosceles triangles. Similar triangles.

Some useful illustrations and applications of similar triangles are afforded by problems dealing with lines which intersect in points out of reach.

The construction of rectangles, parallelograms, and quadrilaterals from adequate data as to sides, angles, and diagonals.

In certain cases it might be desirable to arrange for students, supplied with a measuring tape, to apply their geometry to some elementary surveying, such as measuring and plotting the floor of a room, or a figure drawn on it to a large scale, or a piece of ground, or by setting out a tennis court, etc.

Illustrations of the propositions relating to the areas of squares, rectangles, parallelograms, triangles, and quadrilaterals. The length of the circumference and the area of a circle determined experimentally.

The subject would become more real to a student if he were to find the areas of some actual surfaces from his own measurements.

The location of points by rectangular co-ordinates.

Verification and illustrations of propositions relating to circles. Applications to such problems as the following:—

To find the centre of a given circle.

To draw the tangent to a circle at any given point on the circumference. To find the point of contact of any tangent, or of two circles which touch each other.

Draw two circles, or a line and circle, to touch; guess the point of contact; do this several times, and in each case find the actual point of contact by construction, so as to show that the guess is unreliable.

To describe a circle to pass through three points or to touch three lines.

To describe a circle of *given radius*, (*a*) to touch a given line or circle at a given point; (*b*) to pass through two points; (*c*) to pass through one point and to touch a line or circle; (*d*) to touch two lines, a line and circle, or two circles.

Here a useful variation of ordinary processes may be introduced by the employment of tracing-paper.

Problems requiring a knowledge of the properties of the angles in a semicircle or in any segment of a circle, and the relations which these angles bear to the angles between the chord and the tangent at one extremity.

Insert two pins in the drawing-paper. Let a set-square, with two of its edges always in contact with the pins, be moved into successive positions, and in each case draw lines along the edges. Note the path traced by the corner of the square. Do this for angles of 90°, 60°, 45°, and 30°. Observe what happens as the corner of the square approaches its limiting position, coinciding with one of the pins. Tracing-paper may be used instead of a set-square.

Simple examples of geometrical loci, tracing-paper being employed whenever convenient.

Miscellaneous simple problems and applications of geometrical principles.

Some useful problems may be introduced to be solved by the help of loci or by successive approximations.

Vectors.—The distinction between *scalar* and *vector* quantities. The representation of a vector quantity—(*a*) graphically, by the directed segment of a line; (*b*) numerically, by the statement of a magnitude and an angle.

Thus a vector **A** may be defined by the expression A_α, as 26_{35°, where A is its magnitude, and α the angle, measured anti-clockwise, which its line, directed outwards from O, makes with a fixed line OX.

Addition of vectors. The vector triangle and the vector polygon. Vector summation applied to linear positions, displacements, and velocities; verification by drawing that the result is the same in whatever sequence the vectors are added. Rectangular components of a vector, with illustrations. The parallelogram of vectors.

In all cases numerical results should be measured from the constructions.

Vector subtraction applied to relative positions, displacements, and velocities.

Vector summation applied to forces acting at a point. Experimental verification of the triangle and polygon of forces by means of cords, pulleys, and weights and graphical construction.

Every student should have an opportunity of actually making this experiment himself, and not merely of seeing the teacher or some one else make it.

Opportunities of introducing and explaining the nomenclature of the subject will occur from time to time during the above course in plane geometry.

SOLID GEOMETRY

The position of a point in space defined by three rectangular co-ordinates, and represented on paper (*a*) by the three projections on the planes of reference; (*b*) by a plan and elevation; (*c*) by a figured plan. Distances of the point from the axes, the origin, and the "*xy*" line.

The student should be directed how to cut and fold his drawing-paper so as to bring the vertical planes of reference or projection into their actual upright positions.

A straight line defined and represented by two points as above. Length of line; inclination to horizontal plane; traces of line.

A plane defined by the lengths of its intercepts on three rectangular axes, and exhibited graphically (*a*) by its traces on the three co-ordinate planes; (*b*) by the horizontal and a vertical trace; (*c*) by a scale of slope. Inclination of plane to horizontal plane. Points and lines lying in plane. Rabattement of plane and of any contained figures.

Again let the student make models to illustrate these problems by cutting and folding his drawing-paper, and thus bringing the plan and elevations into their true relative positions in space.

The cube, prism, and pyramid. Plans and elevations in simple positions. Sections by horizontal, vertical, and inclined planes. Simple developments. Projections of the triangle and square when situated with one edge in the horizontal plane. Alteration of the ground line.

A student may build up models in paper of the cube, prism, and pyramid from their developments.

Metric projection, applying the principle that *parallel lines project into parallel lines, all to the same scale.*

Simple applications of the preceding problems in Solid Geometry requiring quantitative measurements.

The selection of useful applications will be influenced by the class of students under instruction; some may be interested in Machine Construction, others in Building Construction, etc.

In this course of solid geometry, opportunities will occur of explaining the terms in common use, and what is meant by projection, by the angle between a line and a plane, and the angle between two planes, etc.

It would be an advantage if there were one or two fairly large models of lines and planes, in combination, lengths and angles connected with which could be actually measured by the student, using a rule, square, and templets. The models could then be measured and represented in projection, and the same lengths and angles found by methods of Solid Geometry and compared with the previous values.

PRACTICAL PLANE AND SOLID GEOMETRY

Elementary Stage

Copy of Evening Examination Paper, 1903

Note.—The diagrams are reproduced $\frac{1}{3}$ size.

INSTRUCTIONS.—Only *eight* questions are to be answered, *three* of which must be Nos. 1, 10 and 12.

1. *Draw two lines*, OA, OA, *each* 10" *long, and containing an angle* AOA *of* 88°. *Measure accurately and state the distance apart of* A, A. (6)

2. Construct a triangle ABC to the following data :—

$AB = 3.8''$; $BC = 3.3''$; $\sin ABC = 0.652$.

Taking AB as the base, draw the perpendicular from the vertex C, and measure the altitude of the triangle. Verify the result by calculation, using the given sine. What is the area of the triangle?

What is the length of the base of a triangle of equal area, but having an altitude of 1.6"? (14)

***3.** The given circle is divided into two sectors, the areas of which represent the relative areas of land and water on the earth's surface.

If the area of the land is 52 millions of square miles, what is the area of the water? (12)

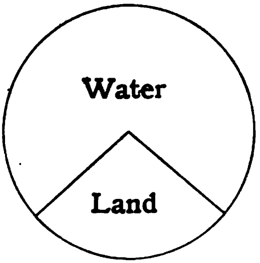
***4.** Set out the figure to the given dimensions; then measure the length of the chord AB.

The answer is required to be correct within 0.02", to ensure which you should employ a special method in order to locate B with precision. (12)

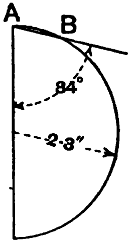
***5.** The two lines LL, LL if produced would meet in a point T out of reach. Through the given point K draw a line which if produced would also pass through T. Find the distance from K to T. (14)

***6.** The given curve is made up of three circular arcs which join tangentially. Plot this figure to the following data :—

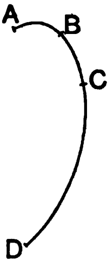
Radius of $AB = 1''$; angle subtended at centre $= 80°$.
Radius of $BC = 2.25''$; length of arc $BC = 1.5''$.
Length of arc $CD = 4''$; angle subtended at centre $= 1$ radian.



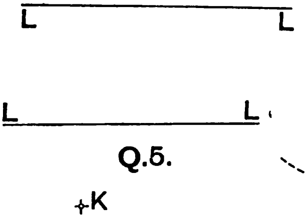
Q.3.



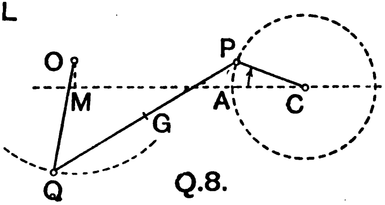
Q.4.



Q.6.



Q.5.



Q.8.

Your figure represents a path to the scale of 1 inch to 100 yards; measure the length of this path. (14)

7. Take the earth to be a sphere of 4000 miles radius. Find and state the distance between two places on the equator which differ in longitude by 1°.

What is the radius of a circle going round the earth's surface in latitude 52°, usually called a parallel of latitude? Find the distance on this circle between two places which differ in longitude by 1°. (14)

*8. PQ is a link, one end, P, of which moves in a circular path, centre C, and the other end, Q, oscillates in a circular arc, centre O. The dimensions are:—

CM = 6′ 0″; MO = 9″; CMO = 90°.
CP = 2′ 0″; PQ = 6′ 3″; OQ = 3′ 0″.
PG = GQ.

Plot the position of G when the angle ACP = 0°, and also when ACP = 45°. Scale 1″ to 1′. (12)

9. A and B are two places 55 miles apart, the direction from A to B being 80° Northwards of East (*i.e.* $AB = 55_{80°}$ miles). How much is B to the East, and how much to the North of A? Adopt a scale of 1″ to 10 miles.

On a certain day in the year the time of sunset at A is 7 hours 56 minutes after noon; and at neighbouring places sunset is 1 minute later for every 10 miles North, and also 1 minute later for every 20 miles West of A. At what time does sunset occur at B on this day? (12)

***10.** *In a laboratory experiment in verification of the polygon of forces,* O *is a small ring to which cords are attached, the pulls in the cords being registered by means of light spring balances* P, Q, R, S. *If the readings of* P *and* Q *were* 1.87 *lbs. and* 1.53 *lbs., as marked on the diagram, what would you expect the readings of* R *and* S *to be?*

Adopt a scale of 1″ *to* 1 *lb.* (12)

***11.** A tramcar is moving at 6 miles an hour as indicated in the diagram, which is a plan. A person entering the car does so along the line AB; what must be his speed along this line if he is to have the same forward motion as the car, *viz.* 6 miles per hour parallel to BC?

In what direction and at what speed would he appear to enter the car to a person seated inside? (14)

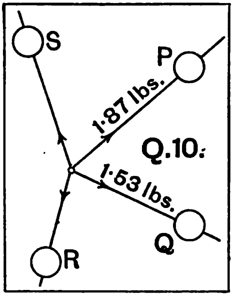
***12.** *The plan is given of a thin* 60° *set square resting on its short edge* BC; *scale* $\frac{1}{2}$.

- (a) *Determine the height of the corner* A *and index its plan* a *in inches.*
- (b) *Determine the length and inclination to the horizontal of the edge* AB.
- (c) *Draw an elevation of the set square on* xy. *Scale* $\frac{1}{2}$ *as in the diagram.* (12)

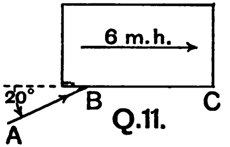
***13.** The plan of the roof of a house is given, scale 1 inch to 10 feet. The surfaces are all inclined at 32° to the horizontal. Find the area in square feet of each portion of the roof, and state the total area. (14)

***14.** The plan of a triangular prism is given, the heights of four of its corners being marked. Write down the heights of the remaining two corners.

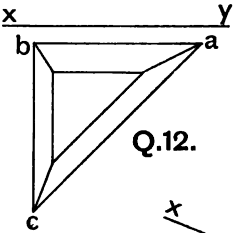
The prism is cut by a vertical plane SS. Draw a sectional elevation on xy, the portion P of the prism in front of the section plane being supposed to be removed. (12)



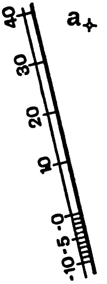
Q.10.



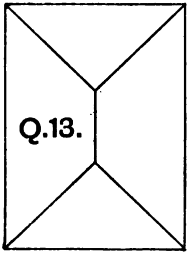
Q.11.



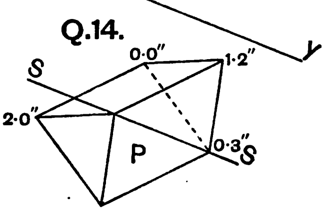
Q.12.



Q.15.



Q.13.



Q.14.

***15.** A plane is represented by a scale of slope, the unit for heights being 0.1". A point A in the plane is shown by its plan a—

(a) Draw the horizontal trace of the plane. Draw also the plan of a horizontal line lying in the plane and containing A.

(b) Measure the height of A and index its plan a.

(c) If the plane were turned into the horizontal plane about its horizontal trace, show where the point A would be carried to, labelling this point A_0. (14)

Copy of Evening Examination Paper, 1904

Stage 1

INSTRUCTIONS.—Answer questions No. 1, No. 8, and No. 11, and *five* others.

1. *Construct a right-angled triangle* ABC *to the following data,* C *being the right angle*:—

Base BC = 10″, *base angle* ABC = 13°.

Measure the height CA. *Obtain tan* 13° *from the tables. Calculate the height* CA. (8)

*2. The following construction is sometimes used for finding the approximate length of a circular arc:—

Let AB be the arc. Draw the chord AB and produce AB to C making BC = $\frac{1}{2}$AB. Draw the tangent at B, and with centre C, radius CA, cut this tangent in D. Then BD = arc BA nearly.

Apply this construction to the given arc. Measure BD in centimetres. Also, by means of tracing paper and a pricker, or otherwise, find the exact length of the arc. Is BD too long or too short, and by what per cent.? (12)

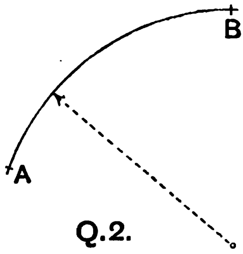
*3. Through the point O draw a line to cut the line CD in E so that the ratio CE : ED shall be equal to the ratio PC : PD. What is this ratio? (12)

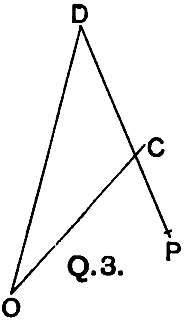
*4. The figure shows a junction of rails for small waggons, the lines drawn dotted being midway between the rails. Set out the figure to a scale of 1 centimetre to 1 foot, working to the given dimensions, and not copying the diagram.

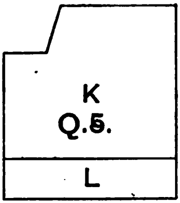
Show the construction for determining the centres of the circular arcs FG, GH which are of equal lengths and radii. Indicate on the drawing the radii to which the several portions of the curved rails must be bent. (14)

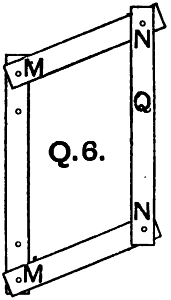
*5. The plan of a hall is given, scale $\frac{1}{2}$ inch to 10 feet; L is the platform and K the body of the hall. Find the area of K in square feet. Calculate the sitting accommodation of K, 30 per cent of its area being occupied by passages, and allowing one person to every 4 square feet of the remainder. (14)

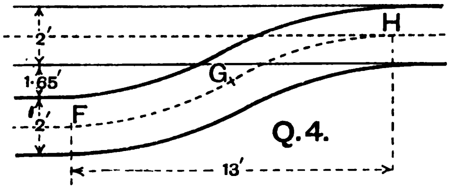
*6. Four strips of sheet celluloid or other material are pinned together at the ends as shown, the whole forming a jointed parallelogram. Suppose the strip MM to be fixed to the drawing board, and let the other strips be turned into successive positions, the two sides MN thus rotating about the points M as centres. Find the locus of Q, the middle point of the strip NN. (12)









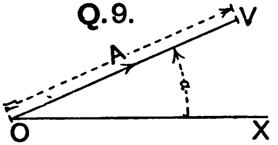


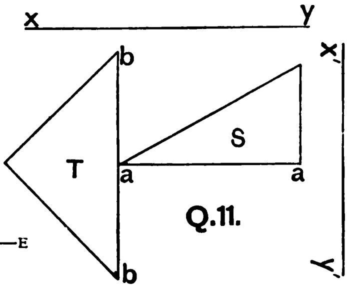
7. A surveyor is making a map on which he wishes to locate two inaccessible objects H and K situated towards the north. He lays off a base line DE, 20 chains or $\frac{1}{4}$ mile long, going due east. When stationed at D, he measures the angles EDK, KDH, by means of a sextant, and finds them to be 51° and 55°. When stationed at E the angles DEH and HEK measure 48° and 62°.

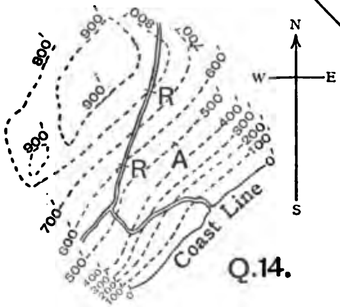
Plot the points D, E, H, K to a scale of 8 inches to the mile. Measure the distance and direction of HK. (14)

8. *A ship* S *is observed from a station* O *to be at a distance of* 3.2 *miles in the direction* 16° *northwards of east; after* $\frac{1}{4}$ *hour it has moved to the position* S′ *distant* 5.6 *miles from* O, *in the direction* 23° N. *of* E. *Plot the points* O, S, S′ *to the scale of* 1 *inch to* 1 *mile. Measure the magnitude and direction of* SS′, *the change of position of the ship. What is the average velocity of the ship during this time?* (10)

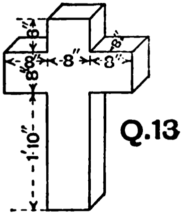
*9. The figure shows a vector $OV = A_\alpha$, A being its magnitude and α its direction measured anti-clockwise from OX. Scale for A, 1 inch to 10 units. Measure A and α.







Scale. ⅔ of an inch to 1 mile.



By construction and measurement determine the component of OV in the direction OX. Verify your answer by calculating A cos α, using the tables to obtain cos α. (12)

10. A load of 2 tons hangs from a crane by a chain 16 feet long. It is required to keep the load 3 feet from the vertical by means of a horizontal rope attached to the bottom of the chain. What force in the rope will be necessary? (12)

***11.** *Two thin set squares* S *and* T *lie in the horizontal plane as shown. Suppose* S *to be turned about* aa *so that its plane is vertical, and* T *to be then turned about* bb *so as to rest against the edge of* S.

Draw the plan of the set squares in their new positions. And draw elevations on xy *and* x'y'. *At what angle is the plane of* T *inclined to the horizontal?* (12)

12. A line AB, 10 cm. long, has its end A in the horizontal plane, and AB is inclined at 35° to the latter. Determine the length of its plan ab, and the height of B. You may calculate these if you wish.

Represent AB by a figured plan. Draw an elevation on a vertical plane which contains B and is 3 cm. from A. (14)

***13.** The dimensioned pictorial view gives particulars of a stone cross. Draw a plan of the cross, scale 1 inch to 1 foot. Project an elevation on a vertical plane making 30° with the front vertical face of the cross. (12)

***14.** A portion of a contoured map of a district is given, showing a road going over the side of a hill. The scale of the map is 2 inches to the mile, and the contour lines are at heights in feet above the sea level as marked. Only so much of the diagram need be copied as is necessary to answer the following questions:—

(*a*) Measure to scale the length of the section of road R R.
(*b*) What is the vertical rise in this length? And at what angle is this portion R R of the road inclined to the horizontal?
(*c*) What would be the direction and slope of a path going straight up the hill and starting from A? (14)

EXAMINATION TABLES

NOTE.—*These tables are from the Directory of the Board of Education, South Kensington. The trigonometrical and logarithmic tables have been revised and rearranged by the Author.*

USEFUL CONSTANTS

1 Inch = 25.40 millimetres.
1 Gallon = .1604 cubic foot = 10 lb. of water at 62° F.
1 Knot = 6080 feet per hour.
Weight of 1 lb. in London = 445,000 dynes.
One pound avoirdupois = 7000 grains = 453.6 grammes.
1 Cubic foot of water weighs 62.3 lb.
1 Cubic foot of air at 0° C. and 1 atmosphere, weighs .0807 lb.
1 Cubic foot of Hydrogen at 0° C. and 1 atmosphere, weighs .00559 lb.
1 Foot-pound = 1.3562×10^7 ergs.
1 Horse-power-hour = 33000 × 60 foot-pounds.
1 Electrical unit = 1000 watt-hours.
Joule's Equivalent to suit Regnault's H, is $\left\{\begin{array}{l}\text{774 ft.-lb.} = \text{1 Fah. unit.} \\ \text{1393 ft.-lb.} = \text{1 Cent. ,,}\end{array}\right.$
1 Horse-power = 33000 foot pounds per minute = 746 watts.
Volts × amperes = watts.
1 Atmosphere = 14.7 lb. per square inch = 2116 lb. per square foot = 760 mm. of mercury = 10^6 dynes per sq. cm. nearly.
A column of water 2.3 feet high corresponds to a pressure of 1 lb. per square inch.
Absolute temp., $t = \theta^\circ$ C. + 273.7°.
Regnault's $H = 606.5 + .305\ \theta^\circ$ C. $= 1082 + .305\ \theta^\circ$ F.

$p\, u^{1.0646} = 479$

$\log_{10} p = 6.1007 - \frac{B}{t} - \frac{C}{t^2},$

where $\log_{10} B = 3.1812$, $\log_{10} C = 5.0882$,

p is in pounds per square inch, t is absolute temperature Centigrade, u is the volume in cubic feet per pound of steam.

$\pi = 3.1416$.

One radian = 57.30 degrees.

To convert common into Napierian logarithms, multiply by 2.3026.

The base of the Napierian logarithms is $e = 2.7183$.

The value of g at London = 32.182 feet per sec. per sec.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	.000	0	0	∞	1	1.414	1.5708	90°
1	.0175	.017	.0175	.0175	57.2900	.9998	1.402	1.5533	89
2	.0349	.035	.0349	.0349	28.6363	.9994	1.389	1.5359	88
3	.0524	.052	.0523	.0524	19.0811	.9986	1.377	1.5184	87
4	.0698	.070	.0698	.0699	14.3006	.9976	1.364	1.5010	86
5	.0873	.087	.0872	.0875	11.4301	.9962	1.351	1.4835	85
6	.1047	.105	.1045	.1051	9.5144	.9945	1.338	1.4661	84
7	.1222	.122	.1219	.1228	8.1443	.9925	1.325	1.4486	83
8	.1396	.140	.1392	.1405	7.1154	.9903	1.312	1.4312	82
9	.1571	.157	.1564	.1584	6.3138	.9877	1.299	1.4137	81
10	.1745	.174	.1736	.1763	5.6713	.9848	1.286	1.3963	80
11	.1920	.192	.1908	.1944	5.1446	.9816	1.272	1.3788	79
12	.2094	.209	.2079	.2126	4.7046	.9781	1.259	1.3614	78
13	.2269	.226	.2250	.2309	4.3315	.9744	1.245	1.3439	77
14	.2443	.244	.2419	.2493	4.0108	.9703	1.231	1.3265	76
15	.2618	.261	.2588	.2679	3.7321	.9659	1.218	1.3090	75
16	.2793	.278	.2756	.2867	3.4874	.9613	1.204	1.2915	74
17	.2967	.296	.2924	.3057	3.2709	.9563	1.190	1.2741	73
18	.3142	.313	.3090	.3249	3.0777	.9511	1.176	1.2566	72
19	.3316	.330	.3256	.3443	2.9042	.9455	1.161	1.2392	71
20	.3491	.347	.3420	.3640	2.7475	.9397	1.147	1.2217	70
21	.3665	.364	.3584	.3839	2.6051	.9336	1.133	1.2043	69
22	.3840	.382	.3746	.4040	2.4751	.9272	1.118	1.1868	68
23	.4014	.399	.3907	.4245	2.3559	.9205	1.104	1.1694	67
24	.4189	.416	.4067	.4452	2.2460	.9135	1.089	1.1519	66
25	.4363	.433	.4226	.4663	2.1445	.9063	1.075	1.1345	65
26	.4538	.450	.4384	.4877	2.0503	.8988	1.060	1.1170	64
27	.4712	.467	.4540	.5095	1.9626	.8910	1.045	1.0996	63
28	.4887	.484	.4695	.5317	1.8807	.8830	1.030	1.0821	62
29	.5061	.501	.4848	.5543	1.8040	.8746	1.015	1.0647	61
30	.5236	.518	.5000	.5774	1.7321	.8660	1.000	1.0472	60
31	.5411	.534	.5150	.6009	1.6643	.8572	.985	1.0297	59
32	.5585	.551	.5299	.6249	1.6003	.8480	.970	1.0123	58
33	.5760	.568	.5446	.6494	1.5399	.8387	.954	.9948	57
34	.5934	.585	.5592	.6745	1.4826	.8290	.939	.9774	56
35	.6109	.601	.5736	.7002	1.4281	.8192	.923	.9599	55
36	.6283	.618	.5878	.7265	1.3764	.8090	.908	.9425	54
37	.6458	.635	.6018	.7536	1.3270	.7986	.892	.9250	53
38	.6632	.651	.6157	.7813	1.2799	.7880	.877	.9076	52
39	.6807	.668	.6293	.8098	1.2349	.7771	.861	.8901	51
40	.6981	.684	.6428	.8391	1.1918	.7660	.845	.8727	50
41	.7156	.700	.6561	.8693	1.1504	.7547	.829	.8552	49
42	.7330	.717	.6691	.9004	1.1106	.7431	.813	.8378	48
43	.7505	733	.6820	.9325	1.0724	.7314	.797	.8203	47
44	.7679	.749	.6947	.9657	1.0355	.7193	.781	.8029	46
45	.7854	.765	.7071	1.0000	1.0000	.7071	.765	.7854	45
			Cosine	Co-tangent	Tangent	Sine	Chord	Radians	Degrees
								Angle	

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

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·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

APPENDIX II

NOTE ON STYLE AND FINISH IN INKED DRAWINGS

IF it be desired to execute an inked drawing or tracing in good style, the following points will require attention :—

1. All the lines should be drawn with the aid of instruments. Freehand drawing is to be avoided.

2. The ink used must be sufficiently dense to give a very black glossy line, and not a line of a weakly brown colour. Bottled ink may be purchased, but for the best work a draughtsman prepares his own ink from the stick.

3. All the lines of a drawing must be of the same uniform width, and should be thick and bold rather than delicate and fine, except for special parts in which fine lines are appropriate, such as section lines, dimension lines, construction lines.

4. Lines should terminate exactly at the points intended, and not be carelessly prolonged or left too short.

5. Circles, lines, or curves which touch should do so accurately. When lines and arcs join tangentially, the place where one ends and the other begins should not be noticeable by any break of continuity at the junction. This result is more readily attained if the circles are first inked in beginning with the larger ones, then the straight lines, and finally any curves which require the use of guiding templates.

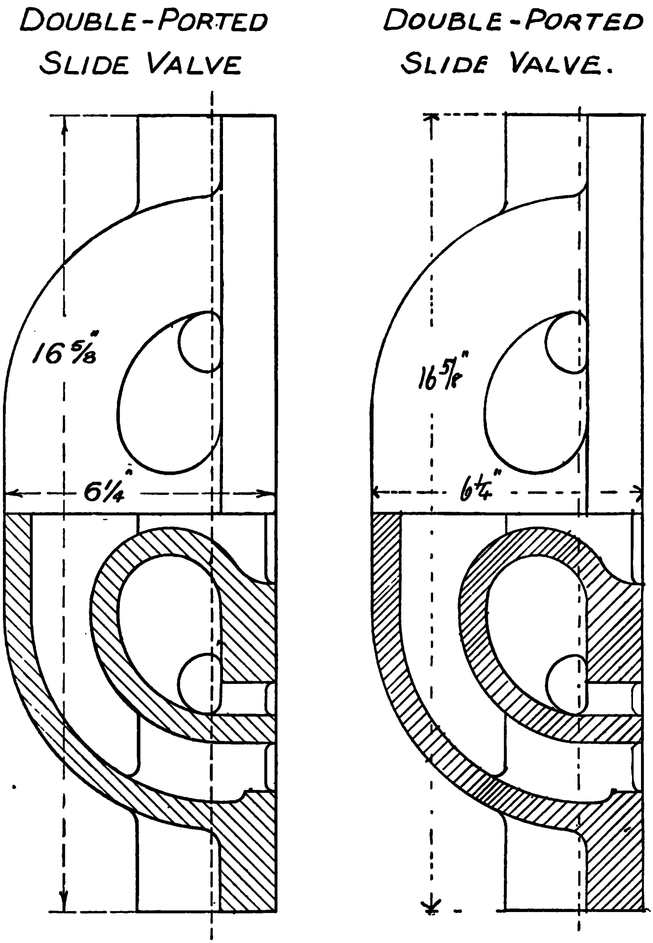
6. Dotted lines should be drawn carefully so as to be of equal lengths and evenly spaced. The spaces may be shorter than the lines.

7. Section lines need not be very close together, but their distance apart must not vary.

8. If parts of the drawing are tinted by flat washes of colour, these must be laid on quite evenly.

9. Cast shadows and graded shading should be scrupulously avoided unless these are correctly projected and well executed, to do which requires an extended preliminary special training. Remember that a drawing, otherwise good, is easily spoiled by immature attempts at shading.

10. For titles, written descriptions, and dimensions, ordinary handwriting is inadmissible. The letters and figures must be well and



CONTRAST BETWEEN
GOOD AND BAD DRAUGHTSMANSHIP.

carefully formed and spaced, and be very legible. Plain block lettering is often used. The arrowheads at the ends of dimension lines should be neat and unobtrusive.

In the accompanying drawings, placed side by side for comparison, some of the points enumerated above are illustrated, both in the breach and in the observance.

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